Math 595 Quantum channels

Exercise sheet 2 – February 14, 2023

Unless stated otherwise, $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \dots$ denote finite-dimensional Hilbert spaces.

1. Let $T\colon \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ and $S\colon \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3)$ be CPTP maps. Show that $S \circ T\colon \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_3)$ is CPTP as well.

2. $\frac{\operatorname{tr}\rho}{2}\mathbb{1} = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$ for any $\rho \in \mathcal{L}(\mathbb{C}^2)$.
   Hint: You can show this by brute-force computation; a more elegant proof is based on Schur's Lemma.

3. Let $\vartheta\colon X \mapsto X^T$ denote the transition map. Show that $\vartheta \circ T \circ \vartheta$ is CP iff T is CP.

4. Any linear operator $W \in \mathcal{L}(\mathbb{C}^2)$ can be written as

$$W = \frac{1}{2}(w_0\mathbb{1} + w_1 X + w_2 Y + w_3 Z) \tag{1}$$

   with $w_i \in \mathbb{C}$. Show the following:

   (a) W is Hermitian iff $w_i \in \mathbb{R}$.

   (b) W has unit trace iff $w_0 = 1$.

   (c) W is positive semidefinite iff $\|\mathbf{w}\|_2 \leq w_0$, where $\mathbf{w} = (w_x, w_y, w_z)^T \in \mathbb{R}^3$ is called the *Bloch vector*.

   (d) W is a pure qubit state iff $\|\mathbf{w}\|_2 = 1$.

5. Setting $\sigma = (X, Y, Z)$, a qubit-qubit channel $\mathcal{N}\colon \mathcal{L}(\mathbb{C}^2) \to \mathcal{L}(\mathbb{C}^2)$ can be written with respect to the basis (1) as

$$\mathcal{N}\left(\frac{1}{2}(w_0\mathbb{1} + w_1 X + w_2 Y + w_3 Z)\right) = \frac{1}{2}\left(w_0\mathbb{1} + (\mathbf{t} + T\mathbf{w}) \cdot \sigma\right), \tag{2}$$

   where $\mathbf{t} \in \mathbb{R}^3$ and T is a real 3×3-matrix. Alternatively, one may write the action of $\mathcal{N}$ on the vector $(w_0, w_1, w_2, w_3)^T$ in terms of a 4×4-matrix

$$N = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & T \end{pmatrix}. \tag{3}$$

   The fact that $N_{1j} = 0$ for $j = 2, 3, 4$ reflects the trace-preserving condition for $\mathcal{N}$.
   Show the following:

   (a) $\mathcal{N}$ is unital iff $\mathbf{t} = \mathbf{0}$.

(b) $\mathcal{N}$ is a Pauli channel, $\mathcal{N}: \rho \mapsto p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z$, iff $\mathbf{t} = \mathbf{0}$ and the corresponding matrix T defined via (2) is diagonal.[1]

(c) Use 5b to show that any unital qubit channel $\mathcal{N}$ is unitarily equivalent to a Pauli channel, i.e., there exist unitaries U and V such that $U\mathcal{N}(V \cdot V^\dagger)U^\dagger$ is a Pauli channel.
Hint: Use the singular value decomposition for the matrix T. Linear transformations acting on $\mathbf{w} \in \mathbb{R}^3$ are mapped to linear transformations on $\mathbb{C}^2$ via (2). What is the image of an orthogonal transformation under this mapping?
Comment: Following Footnote 1, the resulting Pauli map is indeed CP if $\mathcal{N}$ is a unital channel.

(d) Use 5c to show that every unital qubit channel $\mathcal{N}$ is a mixed-unitary channel, i.e., there are unitaries $U_1, \ldots, U_k$ and a probability distribution $(p_1, \ldots, p_d)$ such that $\mathcal{N} = \sum_{i=1}^k p_i U_i \cdot U_i^\dagger$.

6. Determine complementary channels of:

   (a) the erasure channel $\mathcal{E}_p: \rho \mapsto (1-p)\rho + p\,\mathrm{tr}(\rho)|e\rangle\langle e|$;

   (b) the Z-dephasing channel $\mathcal{F}_p^Z: \rho \mapsto (1-p)\rho + Z\rho Z$.

7. Recall that a generalized dephasing channel on $\mathbb{C}^d$ is defined in the following way: Let $\{|i\rangle\}_{i=1}^d$ be an arbitrary orthonormal basis for $\mathbb{C}^d$, and let $\{|\phi_i\rangle_E\}_{i=1}^d$ be an *arbitrary* set of vectors in an auxiliary Hilbert space $\mathcal{H}_E$ with $\dim \mathcal{H}_E \geq 2$. Then the channel isometry is defined by the action on this basis as $|i\rangle \mapsto |i\rangle \otimes |\phi_i\rangle_E$.

   (a) Determine a basis $\{|i\rangle\}_{i=0,1}$ and vectors $\{|\phi_i\rangle_E\}_{i=0,1}$ for the Z-dephasing channel in 6b.

   (b) Prove that any complementary channel of a generalized dephasing channel is entanglement-breaking.

   (c) Prove that generalized dephasing channels are degradable.
   Hint: Either explicitly construct a degrading map $\mathcal{D}$ satisfying $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$ where $\mathcal{N}$ is a generalized dephasing channel, or use 7b directly.

   (d) Let $U_1, \ldots, U_k \in \mathcal{U}(\mathbb{C}^d)$ be unitaries, $(p_1, \ldots, p_k)$ be a probability distribution, and $\mathcal{N} = \sum_{i=1}^k p_i U_i \cdot U_i^\dagger$ be a mixed unitary channel. Give a sufficient condition for $\mathcal{N}$ to be a generalized dephasing channel.

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[1] Let $T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}$, then the corresponding Pauli channel has parameters

$$p_0 = \frac{1}{4}(t_1 + t_2 + t_3 + 1) \quad p_1 = \frac{1}{4}(t_1 - t_2 - t_3 + 1) \quad p_2 = \frac{1}{4}(-t_1 + t_2 - t_3 + 1) \quad p_3 = \frac{1}{4}(-t_1 - t_2 + t_3 + 1), \tag{4}$$

which defines a probability distribution if and only if

$$1 \pm t_3 \geq |t_1 \pm t_2|. \tag{5}$$

These relations coincide with the complete positivity conditions for unital maps defined via (2), as proved by King & Ruskai in arXiv:quant-ph/9911079.