Math 595 Quantum channels Exercise sheet 1

Exercise sheet 1 – January 24, 2023

Unless stated otherwise, $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \ldots$ denote finite-dimensional Hilbert spaces.

1. Prove the following statements:

   (a) Every positive semidefinite operator is Hermitian.

   (b) Let X be a Hermitian operator with largest eigenvalue λ_1. Then $X \leq \lambda_1 \mathbb{1}$. In particular, every quantum state ρ satisfies $\rho \leq \mathbb{1}$.
   Note: For Hermitian operators $A, B \in \mathcal{B}(\mathcal{H})$, the Löwner order *is defined as*

   $$A \geq B :\Leftrightarrow (A - B) \geq 0. \tag{1}$$

   (c) If $X, Y \geq 0$, then $X \otimes Y \geq 0$.

   (d) Let $A, B \in \mathcal{B}(\mathcal{H})$ be diagonalizable. Then A and B commute if and only if they are simultaneously diagonalizable: $[A, B] = AB - BA = 0$ iff there exists an invertible $S \in \mathcal{B}(\mathcal{H})$ such that both SAS^{-1} and SBS^{-1} are diagonal.

2. Prove the following "steering-like" identity: For every $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ there is a linear map $K \colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

   $$|\psi\rangle = (\mathbb{1} \otimes K)|\gamma\rangle, \tag{2}$$

   where $|\gamma\rangle = \sum_{i=1}^{\dim \mathcal{H}_1} |i\rangle \otimes |i\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_1$ and $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_1}$ is some orthonormal basis for $\mathcal{H}_1$.

3. Prove the Schmidt decomposition theorem: For any $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ there are non-negative coefficients $\{\lambda_i\}_{i=1}^r$ (called *Schmidt coefficients*) and sets of orthonormal vectors $\{|\alpha_i\rangle\}_{i=1}^r \subset \mathcal{H}_1$ and $\{|\beta_i\rangle\}_{i=1}^r \subset \mathcal{H}_2$ (called *Schmidt vectors*) such that $|\psi\rangle = \sum_{i=1}^r \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle$. The marginals of the pure state ψ on $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by

   $$\rho_1 = \operatorname{tr}_2 \psi = \sum_{i=1}^r \lambda_i^2 \, |\alpha_i\rangle\langle\alpha_i| \qquad\qquad \rho_2 = \operatorname{tr}_1 \psi = \sum_{i=1}^r \lambda_i^2 \, |\beta_i\rangle\langle\beta_i|, \tag{3}$$

   respectively, and these are spectral decompositions. In particular, $\operatorname{rk} \rho_1 = \operatorname{rk} \rho_2 = r$, which is called the *Schmidt rank* of $|\psi\rangle$. A pure state is entangled iff its Schmidt rank is greater than 1.
   Hint: Set $d_i = \dim \mathcal{H}_i$ and let $\{|e_i\rangle\}_{i=1}^{d_1} \subset \mathcal{H}_1$ and $\{|f_j\rangle\}_{j=1}^{d_2} \subset \mathcal{H}_2$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then we can write

   $$|\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} c_{ij} |e_i\rangle \otimes |f_j\rangle$$

for some $c_{ij} \in \mathbb{C}$. Now apply the singular value decomposition to the $(d_1 \times d_2)$-matrix C with components $(C)_{ij} = c_{ij}$, and extract the Schmidt coefficients and Schmidt vectors from it.

4. Let $\rho \in \mathcal{B}(\mathcal{H})$ be a quantum state, and let $|\psi^\rho\rangle \in \mathcal{H} \otimes \mathcal{H}'$ and $|\phi^\rho\rangle \in \mathcal{H} \otimes \mathcal{H}''$ be two purifications of ρ with purifying Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}''$, i.e., $\mathrm{tr}_{\mathcal{H}'}\, \psi^\rho = \rho = \mathrm{tr}_{\mathcal{H}''}\, \phi^\rho$. Assuming without loss of generality that $\dim \mathcal{H}' \leq \dim \mathcal{H}''$, prove that there exists an isometry $V \colon \mathcal{H}' \to \mathcal{H}''$ such that $|\phi^\rho\rangle = (\mathbb{1} \otimes V)|\psi^\rho\rangle$.
   Hint: Use the Schmidt decomposition for bipartite states.

5. Consider the matrix

$$\tau = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}, \quad \text{with } a, b \in \mathbb{C}. \tag{4}$$

   Determine necessary and sufficient conditions on a, b for τ to be a valid quantum state on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. When is τ a pure state?

6. Consider the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

   Let further $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the identity matrix. Show that $\{\mathbb{1}, X, Y, Z\}$ is a basis for the real vector space of Hermitian matrices on $\mathbb{C}^2$.

7. Let $\{|0\rangle, |1\rangle\}$ be an orthonormal basis of $\mathbb{C}^2$ and consider the following vectors on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right) \tag{6}$$

$$|\Phi^-\rangle = (\mathbb{1} \otimes Z)|\Phi^+\rangle \tag{7}$$

$$|\Psi^+\rangle = (\mathbb{1} \otimes X)|\Phi^+\rangle \tag{8}$$

$$|\Psi^-\rangle = (\mathbb{1} \otimes -iY)|\Phi^+\rangle. \tag{9}$$

   Show that $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$ is an orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$.

   Note: This is often called the Bell basis of two qubits. It consists of four orthonormal maximally entangled states, and is used in protocols like quantum teleportation and dense coding.