MATH 595 Quantum channels II:
Data-processing, recovery channels, and quantum Markov chains

Exercise sheet – April 27, 2021

Unless stated otherwise, $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \dots$ denote finite-dimensional Hilbert spaces.

1. We denote by $D(\rho,\sigma) = \frac{1}{2}\|\rho - \sigma\|_1$ the trace distance and by $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$ the fidelity between ρ and σ. In the following, let $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ be pure quantum states. We write $\psi \equiv |\psi\rangle\langle\psi|$ and $\phi \equiv |\phi\rangle\langle\phi|$.

   (a) Show that $F(\psi,\phi) = |\langle\psi|\phi\rangle|$.

   (b) Show that $D(\psi,\phi) = \sqrt{1 - F(\psi,\phi)^2}$.
   Hint: Write $|\phi\rangle = \cos(\theta)|\psi\rangle + \sin(\theta)|\psi^\perp\rangle$ *for some vector* $|\psi^\perp\rangle \in \mathcal{H}$ *with* $\langle\psi^\perp|\psi\rangle = 0$.

2. Let $X \geq 0$. Show that $\langle\psi|X|\psi\rangle = 0$ implies that $X|\psi\rangle = 0$.
   Hint: Use the spectral decomposition of X.

3. Let $\rho, \sigma \in \mathcal{B}(\mathcal{H})$ be quantum states and $\varepsilon \in (0,1)$. Consider quantum hypothesis testing, in which the task is to distinguish between ρ or σ with a measurement. Let this measurement be given as the 2-element POVM $\{T, \mathbb{1} - T\}$ with a test operator $0 \leq T \leq \mathbb{1}$. We assume that T corresponds to inferring that the unknown state is ρ, while $\mathbb{1} - T$ corresponds to the state σ. The type-I error $\alpha(T)$ and the type-II error $\beta(T)$ are defined as

$$\alpha(T) = \mathrm{tr}(\rho(\mathbb{1} - T)) \qquad \text{and} \qquad \beta(T) = \mathrm{tr}(\sigma T).$$

   In asymmetric hypothesis testing, the goal is to minimize the type-II error under the constraint that the type-I error remains bounded, $\alpha(T) \leq \varepsilon$. This is quantified in terms of a quantity called the *hypothesis testing relative entropy* $D_H^\varepsilon(\rho\|\sigma)$, defined as

$$D_H^\varepsilon(\rho\|\sigma) := -\log \min_{0 \leq T \leq \mathbb{1}} \{\mathrm{tr}(\sigma T) \colon\ \mathrm{tr}(\rho T) \geq 1 - \varepsilon\}.$$

   Prove the data-processing inequality for the hypothesis testing relative entropy: For a quantum channel $\mathcal{N}$ and states ρ, σ,

$$D_H^\varepsilon(\rho\|\sigma) \geq D_H^\varepsilon(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)).$$

   Hint: Use a similar strategy as in the proof of Proposition 2 in the lecture, in which we proved the data-processing inequality for the trace distance.

4. Let $X, Y \in \mathcal{B}(\mathcal{H})$ be strictly positive, i.e., $X, Y > 0$. Show the following statements:

   (a) $\operatorname{tr}\log(XY) = \operatorname{tr}\log X + \operatorname{tr}\log Y$.
   Hint: Use the well-known identity $\det\exp X = \exp\operatorname{tr} X$.

   (b) If $[X, Y] = 0$, then $\log(XY) = \log X + \log Y$.

   (c) $\log(X \otimes Y) = \log(X) \otimes \mathbb{1} + \mathbb{1} \otimes \log Y$.

5. A function $f\colon I \to \mathbb{R}$ is called *operator monotone*, if for any Hermitian $A, B \in \mathcal{B}(\mathcal{H})$ with $\operatorname{spec} A, \operatorname{spec} B \subset I \subset \mathbb{R}$ we have $f(A) \leq f(B)$ if $A \leq B$. Use the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

   to show that $f(t) = t^2$ is *not* operator monotone, that is, $A \leq B$ but $A^2 \nleq B^2$.

6. A function $f\colon I \to \mathbb{R}$ is called *operator convex*, if for any Hermitian $A, B \in \mathcal{B}(\mathcal{H})$ with $\operatorname{spec} A, \operatorname{spec} B \subset I \subset \mathbb{R}$ and $\lambda \in [0,1]$ we have $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$. Use the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

   and $\lambda = 1/2$ to show that $f(t) = t^3$ is *not* operator convex.

7. Let $A, B \in \mathcal{B}(\mathcal{H})$ be Hermitian with $A \leq B$.

   (a) Show that $XAX^\dagger \leq XBX^\dagger$ for any $X \in \mathcal{B}(\mathcal{H})$.

   (b) Show that $T \mapsto T^{-1}$ is order-reversing for commuting, strictly positive operators: If $0 < A \leq B$ with $[A, B] = 0$, then $A^{-1} \geq B^{-1}$.

   (c) Use 7a and 7b to show that $t \mapsto -t^{-1}$ is operator monotone.

   (d) Use 7c to show that $t \mapsto -(t+s)^{-1}$ is operator monotone for any $s > 0$.

   (e) Use 7d and the integral representation

$$\log t = \int_0^\infty ds\, (1+s)^{-1} - (t+s)^{-1}$$

   to show that $t \mapsto \log t$ is operator monotone.