Recap

·) Strong subadditivity: $\underbrace{S(AB)+S(BC)-S(B)-S(ABC)}_{=: I(A;C|B)} \geq 0$

$$=: I(A;C|B) \geq 0$$

·) Quantum Markov chains (QMC): $I(A;C|B)=0$

$$\Leftrightarrow \exists R: B \to BC \text{ s.t. } \rho_{ABC} = (id_A \otimes R)(\rho_{AB})$$

·) Structure of quantum Markov chains:

ρ_{ABC} is a QMC iff $\exists$ decomposition $\mathcal{H}_B = \bigoplus_j \mathcal{H}_{L_j} \otimes \mathcal{H}_{R_j}$

s.t. $\rho_{ABC} = \bigoplus_j q_j \, \rho_{AL_j} \otimes \rho_{R_j C}$ for states $\rho_{AL_j} \in \mathcal{B}(\mathcal{H}_{AL_j})$, $\rho_{R_j C} \in \mathcal{B}(\mathcal{H}_{R_j C})$, and a PD $\{q_j\}$.

·) Crucial ingredient: Koashi / Imoto theorem

Let $\rho_1, \ldots, \rho_N$ be quantum states on $\mathcal{H}$.

i) $\exists$ decomposition $\mathcal{H} = \bigoplus_j J_j \otimes K_j$ s.t.

$$\forall k = 1, \ldots, N: \quad \rho_k = \bigoplus_j q_{j|k} \, \rho_{j|k} \otimes \underset{\text{independent of } k.}{\omega_j}$$

ii) $T|_{J_j \otimes K_j} = id_{J_j} \otimes T_j$, $\quad T_j : \mathcal{B}(K_j) \to \mathcal{B}(K_j)$, $T_j(\omega_j) = \omega_j$

Proof idea of the Koashi/Imoto theorem (full proof: arXiv: quant-ph/0304007)

quantum states $\rho_1, \dots, \rho_N \in \mathcal{B}(\mathcal{H})$, $T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$

with $T(\rho_k) = \rho_k$.

Consider the set $\mathcal{F} = \{ T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) : T(\rho_k) = \rho_k \}$

For every $F \in \mathcal{F}$, define a subalgebra $\mathcal{A}_F \subseteq \mathcal{B}(\mathcal{H})$ via

$$\mathcal{A}_F = \{ X \in \mathcal{B}(\mathcal{H}) : F^\dagger(X) = X \}$$
$$= \{ K_i, K_i^\dagger \}_i' \quad \text{if } F = \sum_i K_i \cdot K_i^\dagger$$

commutant of the Kraus operators of F.

Π_j ... projects onto $\mathcal{H}_{L_j} \otimes \mathcal{H}_{R_j}$

Define $\mathcal{A}_0 = \bigcap_{F \in \mathcal{F}} \mathcal{A}_F$

If $|\mathcal{H}| < \infty$, there is a decomposition $\mathcal{H} = \bigoplus_j \mathcal{H}_{L_j} \otimes \mathcal{H}_{R_j}$ s.t.

$$\mathcal{A}_0 = \bigoplus_j \mathcal{B}(\mathcal{H}_{L_j}) \otimes \mathbb{1}_{R_j}$$

and the projection P_0 onto $\mathcal{A}_0$ has the form

$$P_0(\rho) = \bigoplus_j \mathrm{tr}_{R_j}(\Pi_j \rho \Pi_j) \otimes \omega_j$$

states on $\mathcal{H}_{R_j}$.

Robustness of Markov chains

Classical Markov chains:

$$X \to Y \to Z \text{ is MC} :\Leftrightarrow P_{XYZ} = P_{Z|Y} P_{XY}$$

$$\Leftrightarrow I(X;Z|Y) = 0$$

For arbitrary P_{XYZ}, we can express the CMI in terms of the Kullback-Leibler divergence $D(P\|Q) = \sum_i p_i \log \frac{p_i}{q_i}$ as follows:

(compare: $D(\rho\|\sigma) = \mathrm{tr}(\rho(\log\rho - \log\sigma))$)

$$I(X;Z|Y) = D\left(P_{XYZ} \,\|\, P_{Z|Y} P_{XY}\right)$$

$\Rightarrow$ variational formula: $I(X;Z|Y) = \min_{T\ \mathrm{MC}} D\left(P_{XYZ} \,\|\, T_{XYZ}\right)$

ε-approximate Markov chain: $I(X;Z|Y) \leq \varepsilon$

$$\Rightarrow \min_{T\ \mathrm{MC}} D\left(P_{XYZ} \,\|\, T_{XYZ}\right) \leq \varepsilon$$

Pinsker inequality: $\|P - Q\|_1 \leq \sqrt{2D(P\|Q)}$

$\Rightarrow$ a tripartite PD P_{XYZ} with small CMI is close in trace distance to the set of Markov chains.

This notion of robustness in distance (to QMCs) is <u>not</u> true in the quantum setting:

Prop 24 For all $d > 1$ there exists a state ρ_{ABC} with $|A| = |C| = d$ such that:

i) $I(A;C|B) \leq \frac{2}{d-1} \log d \quad (\to 0 \text{ as } d \to \infty)$

ii) $\min_{\sigma_{ABC} \text{ QMC}} \frac{1}{2} \| \rho_{ABC} - \sigma_{ABC} \|_1 \geq \frac{1}{2}$.

<u>Proof:</u> $S_i = \mathbb{C}^d$, $i = 1, \ldots, d$. Define the so-called Slater determinant:

$$|\psi\rangle_{S_1 \ldots S_d} = \frac{1}{\sqrt{d!}} \sum_{\pi \in S_d} \operatorname{sgn}(\pi) \, |\pi(1)\rangle \otimes \ldots \otimes |\pi(d)\rangle \in (\mathbb{C}^d)^{\otimes d}$$

for some fixed basis $\{|i\rangle\}_{i=1}^d$ of $\mathbb{C}^d$.

Set $\rho_{S_1 \ldots S_d} = |\psi\rangle\langle\psi|_{S_1 \ldots S_d}$.

·) chain rule for mutual information: $I(E;FG) = I(E;F|G) + I(E;G)$

$$\Rightarrow \quad I(S_1; S_2 \ldots S_d) = \sum_{k=2}^{d} I(S_1; S_k | S_2 \ldots S_{k-1})_\rho \leq 2 \log d$$

(Prop 14 (i))

$$\Rightarrow \quad \frac{1}{d-1} \sum_{k=2}^{d} \underbrace{I(S_1; S_k | S_2 \ldots S_{k-1})}_{\geq 0} \leq \frac{2}{d-1} \log d$$

$A = S_1,\ C = S_k,\ B = S_2 \ldots S_{k-1}$

$$\Rightarrow \quad \exists k \text{ s.t. } I(S_1; S_k | S_2 \ldots S_{k-1}) \leq \frac{2}{d-1} \log d \quad \Rightarrow \text{i)}$$

remains to be shown: $\min_{\sigma_{ABC} \text{ QMC}} \frac{1}{2} \| \rho_{ABC} - \sigma_{ABC} \|_1 \geq \frac{1}{2}$.

a) σ_{ABC} is a QMC $\Rightarrow$ σ_{AC} is separable

<u>Proof:</u> Prop 23 $\Rightarrow$ $\sigma_{ABC} = \bigoplus_j q_j \, \sigma^j_{AL_j} \otimes \sigma^j_{R_j C}$

$$\Rightarrow \sigma_{AC} = \bigoplus_j q_j \, \sigma^j_A \otimes \sigma^j_C \quad \text{separable.}$$

b) $\frac{1}{2} \| \rho_{AC} - \sigma_{AC} \|_1 \geq \frac{1}{2}$ if σ_{AC} is separable.

$S_i = \mathbb{C}^d$

<u>Proof:</u> $|\psi\rangle = \frac{1}{\sqrt{d!}} \sum_{\pi \in S_d} \operatorname{sgn}(\pi) \, |\pi(1)\rangle \otimes \ldots \otimes |\pi(d)\rangle \in S_1 \ldots S_d$

$\rho_{S_1 \ldots S_d} = |\psi\rangle\langle\psi|_{S_1 \ldots S_d}$: $A = S_1$, $C = S_k$, $B = S_{2 \ldots k-1}$

Claim (by direct calculation): $\rho_{AC} = \rho_{S_1 S_k} = \frac{1}{d(d-1)} \left(\mathbb{1}_{S_1 S_k} - \mathbb{F}_{S_1 S_k} \right)$

so-called 'antisymmetric state'

Recall: antisymmetric subspace

$$\Lambda^2(\mathbb{C}^d) = \{ |\psi\rangle \in (\mathbb{C}^d)^{\otimes 2} : \mathbb{F}|\psi\rangle = -|\psi\rangle \}$$

Projector onto $\Lambda^2(\mathbb{C}^d)$: $\Pi_a = \frac{1}{2}(\mathbb{1} - \mathbb{F})$

$\dim \Lambda^2(\mathbb{C}^d) = \frac{d(d-1)}{2}$

$$\varrho_{AC} = \varrho_{S_1S_2} = \frac{1}{d(d-1)} \left(\mathbb{1}_{S_1S_2} - \mathbb{F}_{S_1S_2} \right)$$

$$\frac{1}{2} \| \varrho_{AC} - \underset{\substack{\uparrow \\ \in SEP}}{\sigma_{AC}} \|_1 = \max_{0 \le \Lambda \le \mathbb{1}} \mathrm{tr}\, \Lambda \left(\varrho_{AC} - \sigma_{AC} \right) \quad \text{by Lemma 1}$$

choose: $\Pi_s = \frac{1}{2} (\mathbb{1} + \mathbb{F})$ projector onto symmetric subspace.

$$\Rightarrow \frac{1}{2} \| \varrho_{AC} - \sigma_{AC} \|_1 \ge \left| \mathrm{tr}\, \Pi_s \left(\varrho_{AC} - \sigma_{AC} \right) \right|$$

$$= \left| \mathrm{tr}\, \Pi_s \varrho_{AC} - \mathrm{tr}\, \Pi_s \sigma_{AC} \right|$$

$$\mathrm{tr}\, \Pi_s \varrho_{AC} = \mathrm{tr} \left(\frac{1}{2} (\mathbb{1} + \mathbb{F}) \frac{1}{d(d-1)} (\mathbb{1} - \mathbb{F}) \right) = 0 \quad \left(\text{use } \mathbb{F}^2 = \mathbb{1} \right)$$

$$\mathrm{tr}\, \Pi_s \sigma_{AC} = \underbrace{\frac{1}{2} \mathrm{tr}\, \sigma_{AC}}_{= 1/2} + \frac{1}{2} \mathrm{tr}\, \mathbb{F} \sigma_{AC} = \frac{1}{2} + \frac{1}{2} \mathrm{tr}\, \mathbb{F} \sigma_{AC} \ge \frac{1}{2}$$

σ_{AC} sep.: $\quad \sigma_{AC} = \sum_i p_i \, |\omega^i\rangle\langle\omega^i|_A \otimes |\tau^i\rangle\langle\tau^i|_C$

$$\mathrm{tr}\, \mathbb{F} \sigma_{AC} = \sum_i p_i \, \mathrm{tr} \left(\mathbb{F} \left(|\omega^i\rangle\langle\omega^i|_A \otimes |\tau^i\rangle\langle\tau^i|_C \right) \right)$$

$$= \sum_i p_i \, \mathrm{tr} \left(|\tau^i\rangle\langle\omega^i|_A \otimes |\omega^i\rangle\langle\tau^i|_C \right)$$

$$= \sum_i p_i \, |\langle\tau^i|\omega^i\rangle|^2 \ge 0$$

$\Rightarrow \frac{1}{2} \| \varrho_{AC} - \sigma_{AC} \|_1 \ge \frac{1}{2}$ if σ_{AC} is separable.

by a) and b): for any QMC σ_{ABC},

$$\frac{1}{2}\|\rho_{ABC}-\sigma_{ABC}\|_1 \underset{\substack{\text{Prop 2}\\ \text{(DPI for } \|.\|_1)}}{\geq} \frac{1}{2}\|\rho_{AC}-\sigma_{AC}\|_1 \geq \frac{1}{2}$$

$\Rightarrow$ ii) $\square$

Approximate QMCs:

Recall: ρ_{ABC} QMC $\iff \exists\, R: B\to BC$ s.t. $\rho_{ABC}=(id_A\otimes R)(\rho_{AB})$

$\Updownarrow$

$I(A;C|B)=0$

approximate version of this does hold.

Many different statements of this fact, here we mention one proved by Mark M. Wilde in arXiv:1505.04661:

·) Petz recovery map: $\underset{\rho_{BC}}{R}(X_B)=\rho_{BC}^{1/2}\left(\rho_B^{-1/2}X_B\,\rho_B^{-1/2}\otimes \mathbb{1}_C\right)\rho_{BC}^{1/2}$
for ρ_{ABC}

·) for $t\in\mathbb{R}$: $\mathcal{U}_{\omega,t}=\omega^{it}\,.\,\omega^{-it}$

$\to$ "rotated" Petz recovery map: $R_{\rho_{BC},t}=\mathcal{U}_{\rho_{BC},t}\circ R_{\rho_{BC}}\circ\mathcal{U}_{\rho_B,-t}$

Thm (Wilde) $$I(A;C|B)_\rho \geq -\log \sup_{t\in\mathbb{R}} F(\rho_{ABC}, R_{\rho_{BC},t}(\rho_{AB})) \quad (*)$$

$$\geq 0$$

where $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$.

·) Since $0 \leq F(\rho,\sigma) \leq 1$, the RHS in $(*)$ is always ≥ 0.

·) If $I(A;C|B)_\rho \leq \varepsilon$, then by $(*)$ $\exists\, t\in\mathbb{R}$ s.t.

$$F(\rho_{ABC}, R_{\rho_{BC},t}(\rho_{AB})) \geq 2^{-\varepsilon} \approx 1 - \ln(2)\,\varepsilon + O(\varepsilon^2)$$

·) Approximate recovery $\longleftrightarrow$ strengthening of SSA

Open question: does this inequality hold for $t=0$, i.e. for the Petz recovery map?

Most general result for approximate recovery:

Junge et al., arXiv: 1509.07127

Review of QMCs: David Sutter's PhD thesis, arXiv: 1802.05477