Recap

- Holevo bound: classical information $X \sim p_X$ is encoded in a quantum system by choosing quantum states $\{\rho_B^x\}_x$

- Quantum state ensemble $\{p_x, \rho_B^x\}_x$ with corresponding c-q state

$$\rho_{XB} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_B^x$$

- Accessible information: How much information can Bob retrieve from ρ_{XB} using a measurement?

- POVM $M = \{M_y\}_y \longrightarrow p_{y|x} = \mathrm{tr}(M_y \rho_B^x) \longrightarrow$ RV Y

where $(X,Y) \sim p_{XY} = p_{Y|X}\, p_X$

$$I_{acc}(\{p_x, \rho_B^x\}) = \max_{M \text{ POVM}} I(X;Y)$$

- Holevo bound: $I_{acc}(\{p_x, \rho_B^x\}) \leq \chi(\{p_x, \rho_B^x\}) = S\left(\sum_x p_x \rho_B^x\right) - \sum_x p_x S(\rho_B^x)$

- Equality in HB: $I_{acc}(\{p_x, \rho_B^x\}) = \chi(\{p_x, \rho_B^x\})$ iff $[\rho_B^x, \rho_B^{x'}] = 0 \quad \forall x, x'$

- Proof idea: use equality condition for data-processing inequality

The structure of quantum Markov chains

Strong subadditivity: $\forall \rho_{ABC}$, $S(AB)+S(BC)-S(B)-S(ABC)\geq 0$

$$\Leftrightarrow \quad I(A;C|B)\geq 0 \quad \forall \rho_{ABC}$$

Quantum Markov chains: $I(A;C|B)=0$

$$\Leftrightarrow \quad \exists\, R: B\to BC \text{ s.t. } \rho_{ABC}=(id_A\otimes R)(\rho_{AB})$$

Explicit recovery map: Petz recovery map

$$R(X_B)=\rho_{BC}^{1/2}\left(\rho_B^{-1/2}X_B\,\rho_B^{-1/2}\otimes \mathbb{1}_C\right)\rho_{BC}^{1/2}$$

(QMC)

What does the recovery property imply for the structure of quantum Markov chains?

$\boxed{\text{Prop 23}}$ ρ_{ABC} is a QMC iff there exists a direct sum decomposition

$$\mathcal{H}_B=\bigoplus_j \mathcal{H}_{L_j}\otimes\mathcal{H}_{R_j} \quad \text{of } \mathcal{H}_B \text{ such that}$$

$$\rho_{ABC}=\bigoplus_j q_j\,\rho_{AL_j}\otimes\rho_{R_jC}\,, \quad \text{where}$$

- ρ_{AL_j}, ρ_{R_jC} are quantum states $\forall j$
- $\{q_j\}$ is a probability distribution.

Proof: ($\Leftarrow$) Claim: Let $\sigma = \bigoplus_j p_j \sigma_j$, then $S(\sigma) = H(\{p_j\}) + \sum_j p_j S(\sigma_j)$

(p_j: prob. dist., σ_j: states)

Proof: use def. of entropy: $S(\rho) = -\operatorname{tr} \rho \log \rho$

$q = \{q_j\}$

if $\rho_{ABC} = \bigoplus_j q_j \rho_{AL_j} \otimes \rho_{R_jC}$, then $I(A;C|B) = 0$:

$$\mathcal{H}_B = \bigoplus_j \mathcal{H}_{L_j} \otimes \mathcal{H}_{R_j}$$

$$\left.\begin{aligned}
S(AB) &= H(q) + \sum_j q_j \left(S(AL_j) + S(R_j)\right) \\
S(BC) &= H(q) + \sum_j q_j \left(S(L_j) + S(R_jC)\right) \\
S(B) &= H(q) + \sum_j q_j \left(S(L_j) + S(R_j)\right) \\
S(ABC) &= H(q) + \sum_j q_j \left(S(AL_j) + S(R_jC)\right)
\end{aligned}\right\} \begin{aligned} I(A;C|B) &= \\ S(AB)&+S(BC)-S(B)-S(ABC) \\ &= 0 \end{aligned}$$

($\Rightarrow$) We are going to use the following result:

$\boxed{\text{Thm (Koashi / Imoto)}}$

Let $\rho_1, \dots, \rho_N$ be quantum states on a Hilbert space $\mathcal{H}$ ($|\mathcal{H}| < \infty$).

i) $\exists$ direct sum decomposition $\mathcal{H} = \bigoplus_j J_j \otimes K_j$ such that

$$\rho_k = \bigoplus_j q_{j|k}\, \rho_{j|k} \otimes \omega_j$$

($q_{j|k}$: cond. PD; $\rho_{j|k}$: states on J_j; ω_j: fixed states on K_j (independent of k))

Thm (Koashi/Imoto)

Let $\rho_1, \dots, \rho_N$ be quantum states on a Hilbert space $\mathcal{H}$ ($|\mathcal{H}| < \infty$).

i) $\exists$ direct sum decomposition $\mathcal{H} = \bigoplus_j \mathcal{J}_j \otimes \mathcal{K}_j$ such that

$$\rho_k = \bigoplus_j \underbrace{q_{j|k}}_{\text{cond. PD}} \underbrace{\rho_{j|k}}_{\text{states on } \mathcal{J}_j} \otimes \underbrace{\omega_j}_{\text{fixed states on } \mathcal{K}_j \text{ (independent of } k)}$$

ii) For every quantum channel $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with $T(\rho_k) = \rho_k$ for all $k = 1, \dots, N$, there exists a Stinespring unitary

$U = \bigoplus_j \mathbb{1}_{\mathcal{J}_j} \otimes \underbrace{U_{\mathcal{K}_j E}}_{\text{common environment } E \text{ (of } T)}$ with unitaries $U_{\mathcal{K}_j E}$ s.t.

$$\operatorname{tr}_E \Big(U_{\mathcal{K}_j E} \big(\omega_j \otimes \underbrace{\varepsilon}_{\text{fixed environment state}} \big) U_{\mathcal{K}_j E}^{\dagger} \Big) = \omega_j \quad \forall j.$$

$$\Leftrightarrow \; T\big|_{\mathcal{B}(\mathcal{J}_j \otimes \mathcal{K}_j)} = id_{\mathcal{J}_j} \otimes T_j \, , \quad T_j : \mathcal{B}(\mathcal{K}_j) \to \mathcal{B}(\mathcal{K}_j), \; T_j(\omega_j) = \omega_j .$$

We now apply Koashi/Imoto to QMCs:

$\rho_{ABC} = (id_A \otimes R)(\rho_{AB})$ where $R : B \to BC$ is the Petz recovery channel.

$\Rightarrow \rho_{AB} = (id \otimes T)(\rho_{AB})$ with $T = \operatorname{tr}_C \circ R$

QMC: $\rho_{ABC} = (id_A \otimes R)(\rho_{AB})$ where $R: B \to BC$ is the Petz recovery channel.

$\Rightarrow \quad \rho_{AB} = (id \otimes T)(\rho_{AB})$ with $T = tr_C \circ R : B \to B$

This relation holds for a whole <u>ensemble</u> of states:

Let $0 \le M_A \le \mathbb{1}_A$, and define $\mu_B = \frac{1}{p} tr_A\left(\rho_{AB}(M_A \otimes \mathbb{1}_B)\right) \ge 0$

with $p = tr\left(\rho_{AB}(M_A \otimes \mathbb{1}_B)\right)$

Claim: $T(\mu_B) = \mu_B$ for all $0 \le M_A \le \mathbb{1}_A$: $\quad \left(\mu_B = \mu_B(M_A)\right)$

$$T(\mu_B) = \frac{1}{p} T\left(tr_A\left(\rho_{AB}(M_A \otimes \mathbb{1}_B)\right)\right)$$

$$= \frac{1}{p} tr_A\left(\underbrace{id_A \otimes T(\rho_{AB})}_{= \rho_{AB}}(M_A \otimes \mathbb{1}_B)\right) = \mu_B$$

$T = tr_C \circ R : B \to B$

$\Rightarrow$ apply Koashi / Imoto Thm to $\{\mu_B(M_A) : 0 \le M_A \le \mathbb{1}_A\}$:

$\exists$ decomp. $\mathcal{H}_B = \bigoplus_j \mathcal{H}_{L_j} \otimes \mathcal{H}_{R_j}$ s.t.

$$\mu_B = \bigoplus_j q_j(\mu)\, \rho_j(\mu) \otimes \omega_j$$

$\to$ independent of μ

with $\rho_j(\mu) \in B(\mathcal{H}_{L_j})$, $\omega_j \in B(\mathcal{H}_{R_j})$,

$\{q_j(\mu)\}_j$ a prob. dist. for each μ.

Let Π_j be the projector onto $\mathcal{H}_{L_j} \otimes \mathcal{H}_{R_j}$, then $P_0(\mu_B) = \mu_B$ with

$$P_0(\xi) = \bigoplus_j \mathrm{tr}_{R_j}(\Pi_j \xi \Pi_j) \otimes \omega_j$$

to show: $\rho_{AB} = (id_A \otimes P_0)(\rho_{AB})$:

let $0 \leq M_A \leq 1\!\!1_A$, $0 \leq N_B \leq 1\!\!1_B$:

$$\begin{aligned}
\mathrm{tr}\big(\rho_{AB}\,(M_A \otimes N_B)\big) &= p\,\mathrm{tr}\big(\overset{P_0(\mu_B)}{\overset{\shortparallel}{\mu_B}}\, N_B\big) \\
&= p\,\mathrm{tr}\big(P_0(\mu_B)\, N_B\big) \\
&= p\,\mathrm{tr}\big(\mu_B\, P_0^{+}(N_B)\big) \\
&= \mathrm{tr}\big(\rho_{AB}\,(M_A \otimes P_0^{+}(N_B))\big) \\
&= \mathrm{tr}\big(id_A \otimes P_0(\rho_{AB})\,(M_A \otimes N_B)\big)
\end{aligned}$$

$\forall\ 0 \leq M_A \leq 1\!\!1_A,\ 0 \leq N_B \leq 1\!\!1_B$

$\Rightarrow$ by linearity, this holds for all operators σ_{AB}

(choose Hermitian tensor op basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ and recall that every Hermitian X_{AB} can be written as $X_{AB} = X^+_{AB} - X^-_{AB}$ with $X^+, X^- \geq 0$.)

$\Rightarrow \operatorname{tr}(\rho_{AB}\,\sigma_{AB}) = \operatorname{tr}\big((id_A \otimes P_0)(\rho_{AB})\,\sigma_{AB}\big) \quad \forall\, \sigma_{AB}$

$\Rightarrow \rho_{AB} = (id_A \otimes P_0)(\rho_{AB}) \qquad \Big(P_0(\xi) = \bigoplus_j \operatorname{tr}_{R_j}(\Pi_j \xi \Pi_j) \otimes \omega_j\Big)$

$$= \bigoplus_j \underbrace{\operatorname{tr}_{R_j}\big((\mathbb{1}_A \otimes \Pi_j)\,\rho_{AB}\,(\mathbb{1}_A \otimes \Pi_j)\big)}_{=:\, q_j\, \rho_{AL_j}} \otimes \omega_j$$

$$= \bigoplus_j q_j\, \rho_{AL_j} \otimes \omega_j$$

To derive statement about ρ_{ABC}, we apply $R: B \to BC$ to ρ_{AB}:

use (ii) of Koashi/Imoto with $T = \operatorname{tr}_C \circ R$:

Let $U: \mathcal{H}_{BCE} \to \mathcal{H}_{BCE}$ be a unitary, $|\psi\rangle_C$, $|\varphi\rangle_E$ states s.t.

$$R(\tau_B) = \operatorname{tr}_E\big(U(\tau_B \otimes |\psi\rangle\langle\psi|_C \otimes |\varphi\rangle\langle\varphi|_E)U^\dagger\big)$$

$\Rightarrow U = \bigoplus_j \mathbb{1}_{L_j} \otimes U_j \quad$ where U_j is a unitary on $\mathcal{H}_{R_j} \otimes \mathcal{H}_C \otimes \mathcal{H}_E$

$\Rightarrow \rho_{ABC} = (id_A \otimes R)(\rho_{AB})$

$$= \operatorname{tr}_E\Big((\mathbb{1}_A \otimes U)(\rho_{AB} \otimes |\psi\rangle\langle\psi|_C \otimes |\varphi\rangle\langle\varphi|_E)(\mathbb{1}_A \otimes U^\dagger)\Big)$$

$$= \bigoplus_j q_j\, \rho_{AL_j} \otimes \underbrace{\operatorname{tr}_E\big(U_j(\omega_j \otimes |\psi\rangle\langle\psi|_C \otimes |\varphi\rangle\langle\varphi|_E)U_j^\dagger\big)}_{=:\, \rho_{R_jC}} \qquad \square$$