$\boxed{\text{Recap}}$ Proof of data processing inequality and equality conditions

·) $D(\rho_{AB} \| \sigma_{AB}) \geq D(\rho_A \| \sigma_A)$ $\quad$ $(\operatorname{supp} \rho_{AB} \subseteq \operatorname{supp} \sigma_{AB})$

·) assume ρ_{AB} invertible $\Rightarrow$ ρ_A invertible, and σ_{AB}, σ_A invertible

·) define relative modular operators: $\Delta_{AB} = \sigma_{AB} \cdot \rho_{AB}^{-1} : B(\mathcal{H}_{AB}) \to B(\mathcal{H}_{AB})$

$$\Delta_A = \sigma_A \cdot \rho_A^{-1} : B(\mathcal{H}_A) \to B(\mathcal{H}_A)$$

·) $X \in \{AB, A\}$: $\quad \log(\Delta_X)(W_X) = (\log \sigma_X) W_X - W_X \log \rho_X$

·) $D(\rho_X \| \sigma_X) = \langle \rho_X^{1/2}, -\log(\Delta_X)(\rho_X^{1/2}) \rangle$ for $X \in \{AB, A\}$.

·) Goal: find an isometry $\mathcal{U}: A \to AB$ s.t. $\quad (\mathcal{U}^\dagger \mathcal{U} = id_A)$

$$\mathcal{U}(\rho_A^{1/2}) = \rho_{AB}^{1/2} \quad \text{and} \quad \mathcal{U}^\dagger \Delta_{AB} \mathcal{U} = \Delta_A$$

$$\mathcal{U}(X_A) = (X_A \rho_A^{-1/2} \otimes \mathbb{1}_B)\, \rho_{AB}^{1/2}$$

·) $-\log x$ is operator convex:

(f op. conv. if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$)

$$-\log(\Delta_A) = -\log(\mathcal{U}^\dagger \Delta_{AB} \mathcal{U}) \leq -\mathcal{U}^\dagger \log(\Delta_{AB}) \mathcal{U}$$

by the operator Jensen inequality.

·) $D(\rho_A \| \sigma_A) = \langle \rho_A^{1/2}, -\log(\Delta_A)(\rho_A^{1/2}) \rangle \leq \langle \rho_A^{1/2}, -\mathcal{U}^\dagger \log(\Delta_{AB}) \mathcal{U}(\rho_A^{1/2}) \rangle$

$= D(\rho_{AB} \| \sigma_{AB}) \quad$ (DPI)

·) equality in DPI:

$$(*) \quad \langle \rho_A^{1/2}, -\underline{\log}(\mathcal{U}^\dagger \Delta_{AB} \mathcal{U})(\rho_A^{1/2})\rangle = \langle \rho_A^{1/2}, -\mathcal{U}^\dagger \underline{\log}(\Delta_{AB})\mathcal{U}(\rho_A^{1/2})\rangle$$

·) Integral representation: $-\log x = \int_0^\infty dt \left[(x+t)^{-1} - (1+t)^{-1}\right]$

$$\cdot)\ (*) \Rightarrow \int_0^\infty dt \, \langle \rho_A^{1/2}, \underbrace{\left[\mathcal{U}^\dagger(\Delta_{AB}+t)^{-1}\mathcal{U} - (\mathcal{U}^\dagger \Delta_{AB}\mathcal{U}+t)^{-1}\right]}_{=: X_t}(\rho_A^{1/2})\rangle = 0$$

·) $x \mapsto (x+t)^{-1}$ is operator convex: $X_t \geq 0 \ \forall t$ by operator Jensen ineq.

$$\Rightarrow \langle \rho_A^{1/2}, X_t(\rho_A^{1/2})\rangle \geq 0 \quad \forall t$$

·) $\int_0^\infty dt \, \langle \rho_A^{1/2}, X_t(\rho_A^{1/2})\rangle = 0 \quad \Rightarrow \langle \rho_A^{1/2}, X_t(\rho_A^{1/2})\rangle = 0 \quad \forall t$

·) Simple observation: $R \geq 0$, $\langle w | R | w\rangle = 0 \Rightarrow R|w\rangle = 0$

$$\Rightarrow X_t(\rho_A^{1/2}) = 0 \quad \text{or}$$

$$\mathcal{U}^\dagger(\Delta_{AB}+t)^{-1}\underbrace{\mathcal{U}(\rho_A^{1/2})}_{\rho_{AB}^{1/2}} = (\Delta_A + t)^{-1}(\rho_A^{1/2}) \quad (**)$$

·) to show: $(**) \Rightarrow \rho_{AB} = \sigma_{AB}^{1/2}(\sigma_A^{-1/2}\rho_A\sigma_A^{-1/2} \otimes 1_B)\sigma_{AB}^{1/2}$

·) from $(**)$: $\mathcal{U}\mathcal{U}^\dagger(\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) = \mathcal{U}(\Delta_A+t)^{-1}(\rho_A^{1/2})$

·) to show: $(\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) \in \operatorname{Im}\mathcal{U}$

·) leads to: $(\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) = \mathcal{U}\,(\Delta_A+t)^{-1}(\rho_A^{1/2})$

·) $f(x) = x^{-1/2} = C\int_0^\infty dt\,\frac{t^{-1/2}}{x+t}$

$$\Rightarrow \quad \Delta_{AB}^{-1/2}(\rho_{AB}^{1/2}) = \mathcal{U}\,\Delta_A^{-1/2}(\rho_A^{1/2})$$

$$\Rightarrow \quad \rho_{AB} = \sigma_{AB}^{1/2}\left(\sigma_A^{-1/2}\rho_A\sigma_A^{-1/2}\otimes \mathbb{1}_B\right)\sigma_{AB}^{1/2} \qquad \square$$

$\boxed{\text{Thm 18}}$ ρ_{ABC} is a quantum Markov chain iff $I(A;C|B)=0$.

$\exists\, R: B\to BC$ s.t.

$$\rho_{ABC} = (id_A\otimes R)(\rho_{AB})$$

Proof: $\Rightarrow$ $D(\rho_{ABC}\,\|\,\rho_A\otimes\rho_{BC}) \geq D(\rho_{AB}\,\|\,\rho_A\otimes\rho_B)$

DPI: $\nearrow$ wrt. tr_C ($\Leftrightarrow I(A;C|B)\geq 0$)

$\downarrow$ wrt. $id_A\otimes R$

$$\geq D\big((id_A\otimes R)(\rho_{AB})\,\|\,\rho_A\otimes R(\rho_B)\big)$$

$$= D(\rho_{ABC}\,\|\,\rho_A\otimes\rho_{BC})$$

$\Rightarrow I(A;C|B) = 0$

Proof: $\Rightarrow$ $D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) \geq D(\rho_{AB} \| \rho_A \otimes \rho_B)$

$\uparrow$ DPI wrt. tr_C $\qquad (\Leftrightarrow \underline{I(A;C|B) \geq 0})$

$$\geq D\big((\mathrm{id}_A \otimes R)(\rho_{AB}) \,\|\, \rho_A \otimes R(\rho_B)\big)$$

$$= D(\rho_{ABC} \| \rho_A \otimes \rho_{BC})$$

$\Rightarrow I(A;C|B) = 0$

$\Leftarrow$ $I(A;C|B) = 0 \Leftrightarrow D(\rho_{ABC} \| \underline{\rho_A \otimes \rho_{BC}}) = D(\rho_{AB} \| \underline{\rho_A \otimes \rho_B})$

Prop 19: $\Leftrightarrow$

$$\underline{R_{\rho_A \otimes \rho_{BC}}}(X_{AB}) = (\cancel{\rho_A^{1/2}} \otimes \rho_{BC}^{1/2})(\cancel{\rho_A^{-1/2}} \otimes \rho_B^{-1/2})\, X_{AB}\, (\cancel{\rho_A^{-1/2}} \otimes \rho_B^{-1/2}) \otimes \mathbb{1}_C)\,(\cancel{\rho_A^{1/2}} \otimes \rho_{BC}^{1/2})$$

satisfies $\rho_{ABC} = R_{\rho_A \otimes \rho_{BC}}(\rho_{AB})$

$\Rightarrow \tilde{R} : X_B \mapsto \rho_{BC}^{1/2} (\rho_B^{-1/2} X_B \rho_B^{-1/2} \otimes \mathbb{1}_C) \rho_{BC}^{1/2}$

satisfies $\rho_{ABC} = (\mathrm{id}_A \otimes R)(\rho_{AB}) \Rightarrow \rho_{ABC}$ is a quantum Markov chain.

$\square$

Generalize equality condition in DPI to arbitrary quantum chan's

$\boxed{\text{Prop 20}}$ Let $\rho, \sigma \geq 0$, $\operatorname{tr}(\rho) = 1$, $\mathcal{N}: A \to B$ quantum channel
($\operatorname{supp} \rho \subseteq \operatorname{supp} \sigma$)

Then $D(\rho \| \sigma) = D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma))$ iff

the map $R_{\sigma, \mathcal{N}}: B \to A$

$$X_B \mapsto \sigma^{1/2} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{-1/2} X_B \mathcal{N}(\sigma)^{-1/2} \right) \sigma^{1/2}$$

satisfies $R_{\sigma,\mathcal{N}}(\mathcal{N}(\sigma)) = \sigma$ (by construction)

and $R_{\sigma,\mathcal{N}}(\mathcal{N}(\rho)) = \rho$.

Proof: $\Leftarrow$ as in Prop 19, prove first that $R_{\sigma,\mathcal{N}}$ is a quantum chan. on $\operatorname{supp} \mathcal{N}(\sigma)$

·) CP: clear, since $X_B \mapsto \mathcal{N}(\sigma)^{-1/2} X_B \mathcal{N}(\sigma)^{-1/2}$

$$X_B \mapsto \mathcal{N}^\dagger(X_B)$$

$$X_A \mapsto \sigma^{1/2} X_A \sigma^{1/2}$$ are all CP.

·) TP: $\operatorname{tr} R_{\sigma,\mathcal{N}}(X_B) = \operatorname{tr}\left(\sigma \underset{\curvearrowleft}{\mathcal{N}^\dagger} \left(\mathcal{N}(\sigma)^{-1/2} X_B \mathcal{N}(\sigma)^{-1/2}\right)\right)$

$$= \operatorname{tr}\left(\mathcal{N}(\sigma) \mathcal{N}(\sigma)^{-1/2} X_B \mathcal{N}(\sigma)^{-1/2}\right)$$

·) TP: $\operatorname{tr} R_{\sigma,\mathcal{N}}(X_B) = \operatorname{tr}\left(\sigma\, \mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-1/2} X_B\, \mathcal{N}(\sigma)^{-1/2}\right)\right)$

$$= \operatorname{tr}\left(\mathcal{N}(\sigma)\, \mathcal{N}(\sigma)^{-1/2} X_B\, \mathcal{N}(\sigma)^{-1/2}\right)$$

$$= \operatorname{tr}\left(\mathcal{N}(\sigma)^0 X_B\right) = \operatorname{tr} X_B \quad \text{if } \operatorname{supp} X_B \subseteq \operatorname{supp} \mathcal{N}(\sigma)$$

$$\mathcal{N}(\sigma)^0 = \lim_{a \to 0} \mathcal{N}(\sigma)^a = \Pi_{\operatorname{supp} \mathcal{N}(\sigma)}$$

$$\Rightarrow D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq D\left(R_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))\,\|\, R_{\sigma,\mathcal{N}}(\mathcal{N}(\sigma))\right)$$

$$= D(\rho\|\sigma)$$

"$\Leftarrow$" $\mathcal{N}$ quantum channel: $\exists\, \mathcal{H}_E$ and isometry $V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$

s.t. $\mathcal{N}(X_A) = \operatorname{tr}_E V X_A V^\dagger \quad (*)$

by assumption $D(\rho\|\sigma) = D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$

$$D(\rho\|\sigma) = D(V\rho V^\dagger \| V\sigma V^\dagger) \qquad D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \overset{(*)}{=} D(\operatorname{tr}_E V\rho V^\dagger \| \operatorname{tr}_E V\sigma V^\dagger)$$

apply Prop 19: equality holds iff

$$V\rho V^\dagger = (V\sigma V^\dagger)^{1/2}\left(\left(\operatorname{tr}_E V\sigma V^\dagger\right)^{-1/2} \operatorname{tr}_E V\rho V^\dagger \left(\operatorname{tr}_E V\sigma V^\dagger\right)^{-1/2} \otimes \mathbb{1}_E\right)(V\sigma V^\dagger)^{1/2}$$

$$V\rho V^\dagger = \underbrace{(V\sigma V^\dagger)^{1/2}}_{V\sigma^{1/2}V^\dagger}\Big(\underbrace{(\mathrm{tr}_E V\sigma V^\dagger)}_{\mathcal{N}(\sigma)}{}^{-1/2}\underbrace{\mathrm{tr}_E V\rho V^\dagger}_{\mathcal{N}(\rho)}\underbrace{(\mathrm{tr}_E V\sigma V^\dagger)}_{\mathcal{N}(\sigma)}{}^{-1/2}\otimes \mathbb{1}_E\Big)\underbrace{(V\sigma V^\dagger)^{1/2}}_{V\sigma^{1/2}V^\dagger}$$

$$= V\sigma^{1/2}\underbrace{V^\dagger\left(\mathcal{N}(\sigma)^{-1/2}\mathcal{N}(\rho)\mathcal{N}(\sigma)^{-1/2}\otimes\mathbb{1}_E\right)V}_{=\mathcal{N}^\dagger(\dots)}\sigma^{1/2}V^\dagger$$

$$\left[\text{if } \mathcal{N}(.) = \mathrm{tr}_E V . V^\dagger, \text{ then } \mathcal{N}^\dagger(.) = V^\dagger(.\otimes \mathbb{1}_E)V\right]$$

$$\Rightarrow \cancel{V}\rho\cancel{V^\dagger} = \cancel{V}\sigma^{1/2}\mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-1/2}\mathcal{N}(\rho)\mathcal{N}(\sigma)^{-1/2}\right)\sigma^{1/2}\cancel{V^\dagger}$$

$$\Rightarrow \exists\ R_{\sigma,\mathcal{N}}: B \to A,\quad X_B \mapsto \sigma^{1/2}\mathcal{N}^\dagger\left(\mathcal{N}(\sigma)^{-1/2}X_B\,\mathcal{N}(\sigma)^{-1/2}\right)\sigma^{1/2}$$

satisfies $\rho = R_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))$ $\square$

Cor 21 Let ρ,σ be quantum states. Then $D(\rho\|\sigma)\geq 0$, and $D(\rho\|\sigma)=0$ iff $\rho=\sigma$.

Proof: $D(\rho\|\sigma) \geq D(\mathrm{tr}(\rho)\|\mathrm{tr}(\sigma)) = D(1\|1) = 0$.

·) If $\rho = \sigma$, then clearly $D(\rho \| \sigma) = 0$.

·) if $D(\rho\|\sigma) = 0$, then Prop 20 with $\mathcal{N} = \mathrm{tr}$, $\mathcal{N}^{\dagger} = \mathbb{1}$

gives $\rho = \sigma^{1/2} \mathbb{1} \sigma^{1/2} = \sigma$. $\square$