$\boxed{\text{Recap}}$

·) Classical Markov chain $X \to Y \to Z$:

$$P_{XZ|Y} = P_{X|Y} P_{Z|Y} \iff I(X;Z|Y) = 0$$

$$\iff \exists\, W: Y \to Z \text{ with } P_{XYZ} = W_{Z|Y} P_{XY}$$

·) Quantum Markov chain ρ_{ABC}:

$$\exists\, R: B \to BC \text{ s.t. } \rho_{ABC} = (id_A \otimes R)(\rho_{AB}) \qquad (*)$$

·) Thm 18: ρ_{ABC} is quantum Markov chain iff $I(A;C|B) = 0$.

·) "$\Rightarrow$" direction follows using $(*)$ and data-processing.

·) "$\Leftarrow$" direction: Need to understand equality condition in data-processing inequality

·) For Thm 18: suffices to discuss equality for partial trace.

$\boxed{\text{Prop 19}}$ $D(\rho_{AB} \| \sigma_{AB}) = D(\rho_A \| \sigma_A)$

$\iff$ The channel $R_\sigma: A \to AB$, $X_A \mapsto \sigma_{AB}^{1/2} (\sigma_A^{-1/2} X_A \sigma_A^{-1/2} \otimes \mathbb{1}_B) \sigma_{AB}^{1/2}$

satisfies $\rho_{AB} = R_\sigma(\rho_A)$ (we always have $\sigma_{AB} = R_\sigma(\sigma_A)$)

Proof of Prop 19: $(\Leftarrow)$ If $\exists R_\sigma$: $R_\sigma(\rho_A) = \rho_{AB}$, $R_\sigma(\sigma_A) = \sigma_{AB}$

$$\underline{D(\rho_{AB} \| \sigma_{AB})} \underset{\text{tr}_B}{\geq} D(\rho_A \| \sigma_A) \underset{R_\sigma}{\geq} \underline{D(\rho_{AB} \| \sigma_{AB})}$$

DPI

$(\Rightarrow)$ to show:

$$D(\rho_{AB} \| \sigma_{AB}) = D(\rho_A \| \sigma_A) \Rightarrow \exists R_\sigma : A \to AB$$

$$X_A \mapsto \sigma_{AB}^{1/2} (\sigma_A^{-1/2} X_A \sigma_A^{-1/2} \otimes \mathbb{1}_B) \sigma_{AB}^{1/2}$$

s.t. $\rho_{AB} = R_\sigma(\rho_A)$.

$$\Leftrightarrow \rho_{AB} = \sigma_{AB}^{1/2} (\sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \otimes \mathbb{1}_B) \sigma_{AB}^{1/2}$$

we use the following facts:

·) $-\log x$, x^{-1} are op. convex

·) integral representations: $-\log x = \int_0^\infty dt \left[(x+t)^{-1} - (1+t)^{-1} \right]$

$$x^{-1/2} = -\frac{\sin(-\pi/2)}{\pi} \int_0^\infty dt \, \frac{t^{-1/2}}{x+t}$$

Assumption: ρ_{AB} is invertible $(\rho_{AB} > 0)$

·) $\Rightarrow \rho_A$ is invertible $(\operatorname{supp} \rho_{AB} \subseteq \operatorname{supp} \rho_A \otimes \mathcal{H}_B)$

·) by assumption : $\operatorname{supp} \rho_{AB} \subseteq \operatorname{supp} \sigma_{AB} \Rightarrow \sigma_{AB}, \sigma_A$ invertible

Relative modular operators:

$$\Delta_{AB} = \sigma_{AB} \cdot \rho_{AB}^{-1} \qquad \Delta_A = \sigma_A \cdot \rho_A^{-1}$$

Claim: $D(\rho_X \| \sigma_X) = \langle \rho_X^{1/2}, -\log(\Delta_X)(\rho_X^{1/2}) \rangle$

$X \in \{AB, A\}$

Proof: $\log(\Delta_X) = \log(L_{\sigma_X} \circ R_{\rho_X^{-1}})$

$$= \log(L_{\sigma_X}) + \log R_{\rho_X^{-1}}$$

$$= L_{\log \sigma_X} - R_{\log \rho_X} \qquad (\text{Lemma } 7(iv))$$

$$\langle \rho_X^{1/2}, -\log(\Delta_X)(\rho_X^{1/2}) \rangle = -\langle \rho_X^{1/2}, L_{\log \sigma_X}(\rho_X^{1/2}) \rangle$$

$$+ \langle \rho_X^{1/2}, R_{\log \rho_X}(\rho_X^{1/2}) \rangle$$

$$= -\mathrm{tr}(\rho_X^{1/2} \log(\sigma_X) \rho_X^{1/2})$$

$$+ \mathrm{tr}(\rho_X^{1/2} \rho_X^{1/2} \log \rho_X)$$

$$= D(\rho_X \| \sigma_X) \qquad \checkmark$$

Goal: find a map $\mathcal{U}: A \to AB$ s.t.

i) $\mathcal{U}^\dagger \mathcal{U} = id_A$ ii) $\mathcal{U}(\rho_A^{1/2}) = \rho_{AB}^{1/2}$ iii) $\mathcal{U}^\dagger \Delta_{AB} \mathcal{U} = \Delta_A$

Goal: find a map $\mathcal{U}: A \to AB$ s.t.

i) $\mathcal{U}^\dagger \mathcal{U} = id_A$ ii) $\mathcal{U}(\rho_A^{1/2}) = \rho_{AB}^{1/2}$ iii) $\mathcal{U}^\dagger \Delta_{AB} \mathcal{U} = \Delta_A$

The choice $\mathcal{U}(X_A) = (X_A \rho_A^{-1/2} \otimes \mathbb{1}_B) \rho_{AB}^{1/2}$ satisfies i) - iii)

(proof along same lines as before)

$-\log x$ is operator convex: by operator Jensen inequality,

$$-\log(\Delta_A) \overset{iii)}{=} -\log(\mathcal{U}^\dagger \Delta_{AB} \mathcal{U}) \leq -\mathcal{U}^\dagger \log(\Delta_{AB}) \mathcal{U}$$

$$D(\rho_A \| \sigma_A) = \langle \rho_A^{1/2}, -\log(\Delta_A)(\rho_A^{1/2}) \rangle$$

now assume equality

$$\leq \langle \rho_A^{1/2}, -\mathcal{U}^\dagger \log(\Delta_{AB}) \mathcal{U}(\rho_A^{1/2}) \rangle \qquad (*)$$

$$\overset{(ii)}{=} \langle \rho_{AB}^{1/2}, -\log(\Delta_{AB})(\rho_{AB}^{1/2}) \rangle = D(\rho_{AB} \| \sigma_{AB})$$

Integral representation: $-\log x = \int_0^\infty dt \left[\underline{(x+t)^{-1}} - (1+t)^{-1} \right]$

$$X_t := \mathcal{U}^\dagger (\Delta_{AB} + t)^{-1} \mathcal{U} - (\mathcal{U}^\dagger \Delta_{AB} \mathcal{U} + t)^{-1}$$

substituting in $(*)$: $\langle \rho_A^{1/2}, -\log(\mathcal{U}^\dagger \Delta_{AB} \mathcal{U})(\rho_A^{1/2}) \rangle$

$$= \langle \rho_A^{1/2}, -\mathcal{U}^\dagger \log(\Delta_{AB}) \mathcal{U}(\rho_A^{1/2}) \rangle$$

iff $\int_0^\infty dt \langle \rho_A^{1/2}, X_t(\rho_A^{1/2}) \rangle = 0$

substituting in $(*)$: $\langle \rho_A^{1/2}, -\log(\mathcal{U}^+ \Delta_{AB} \mathcal{U})(\rho_A^{1/2})\rangle$

$$= \langle \rho_A^{1/2}, -\mathcal{U}^+ \log(\Delta_{AB})\,\mathcal{U}(\rho_A^{1/2})\rangle$$

iff $\int_0^\infty dt \langle \rho_A^{1/2}, X_t(\rho_A^{1/2})\rangle = 0$

where $X_t = \mathcal{U}^+(\Delta_{AB}+t)^{-1}\mathcal{U} - (\underbrace{\mathcal{U}^+\Delta_{AB}\mathcal{U}}_{\Delta_A}+t)^{-1}$

$x \mapsto (x+t)^{-1}$ is operator convex for $t \geq 0$, and hence by operator Jensen, $X_t \geq 0 \Rightarrow \langle \rho_A^{1/2}, X_t(\rho_A^{1/2})\rangle \geq 0 \;\forall t$

but $\int_0^\infty dt \langle \rho_A^{1/2}, X_t(\rho_A^{1/2})\rangle = 0 \Rightarrow \langle \rho_A^{1/2}, X_t(\rho_A^{1/2})\rangle = 0$ for almost all t

$\Rightarrow \langle \rho_A^{1/2}, X_t(\rho_A^{1/2})\rangle = 0 \;\forall t$ by continuity

$\Rightarrow X_t(\rho_A^{1/2}) = 0$ (general statement: $R \geq 0$ $\langle w|R|w\rangle = 0 \Rightarrow R|w\rangle = 0$)

$\Updownarrow$

$$\mathcal{U}^+(\Delta_{AB}+t)^{-1}(\underbrace{\rho_{AB}^{1/2}}_{\mathcal{U}(\rho_A^{1/2})}) = (\Delta_A+t)^{-1}(\rho_A^{1/2})$$

$$\mathcal{U}^\dagger (\Delta_{AB}+t)^{-1}(\underbrace{\rho_{AB}^{1/2}}_{\mathcal{U}(\rho_A^{1/2})}) = (\Delta_A+t)^{-1}(\rho_A^{1/2}) \qquad (1)$$

$$\Big\downarrow \ \mathcal{U}(.)$$

$$\mathcal{U}\mathcal{U}^\dagger (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) = \mathcal{U}(\Delta_A+t)^{-1}(\rho_A^{1/2}) \qquad (2)$$

Claim: $\mathcal{U}\mathcal{U}^\dagger (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) = (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) \qquad (3)$

$$\Longleftrightarrow \quad (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) \in \operatorname{Im}\mathcal{U}$$

Proof: take square modulus of both sides in (1):

$$\langle \mathcal{U}^\dagger (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}), \mathcal{U}^\dagger (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) \rangle$$

$$\mathcal{U}$$

$$= \langle (\Delta_A+t)^{-1}(\rho_A^{1/2}), (\Delta_A+t)^{-1}(\rho_A^{1/2}) \rangle$$

$$\left((\Delta_A+t)^{-1}\right)^\dagger = (\Delta_A+t)^{-1}$$

$$\langle (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}), \mathcal{U}\mathcal{U}^\dagger (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) \rangle$$

$$= \langle (\Delta_A+t)^{-2}(\rho_A^{1/2}), \rho_A^{1/2} \rangle \qquad (4)$$

take derivative in (1) w.r.t. t:

$$\mathcal{U}^\dagger (\Delta_{AB}+t)^{-2}(\rho_{AB}^{1/2}) = (\Delta_A+t)^{-2}(\rho_A^{1/2})$$

take derivative in (1) w.r.t. t:

$$\mathcal{U}^{\dagger}(\Delta_{AB}+t)^{-2}(\rho_{AB}^{1/2}) = (\Delta_{A}+t)^{-2}(\rho_{A}^{1/2})$$

Substitute in (4):

$$\langle (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}), \mathcal{U}\mathcal{U}^{\dagger}(\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) \rangle$$

$$= \langle \mathcal{U}^{\dagger}(\Delta_{AB}+t)^{-2}(\rho_{AB}^{1/2}), \rho_{A}^{1/2} \rangle$$

$$\mathcal{U}$$

$$= \langle (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}), (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) \rangle$$

same argument as before: $\langle w|\mathcal{U}\mathcal{U}^{\dagger}|w\rangle = \langle w|\mathbb{1}|w\rangle$

and $\mathcal{U}\mathcal{U}^{\dagger} \leq \mathbb{1} \Rightarrow \mathcal{U}\mathcal{U}^{\dagger}|w\rangle = |w\rangle$

or $\mathcal{U}\mathcal{U}^{\dagger}(\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) = (\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2})$ (claim)

Substitute this in (2):

$$(\Delta_{AB}+t)^{-1}(\rho_{AB}^{1/2}) = \mathcal{U}(\Delta_{A}+t)^{-1}(\rho_{A}^{1/2}) \qquad (4)$$

now use integral representation for $x^{-1/2}$: $\quad x^{-1/2} = -\frac{\sin(\pi/2)}{\pi}\int_0^{\infty} dt\, \frac{t^{-1/2}}{x+t}$

with (4) $\Rightarrow$ $\Delta_{AB}^{-1/2}(\rho_{AB}^{1/2}) = \mathcal{U}\,\Delta_{A}^{-1/2}(\rho_{A}^{1/2})$

$$\Delta_{AB}^{-1/2}(\rho_{AB}^{1/2}) = \mathcal{U}\, \Delta_A^{-1/2}(\rho_A^{1/2})$$

LHS: $\Delta_{AB}^{-1/2}(\rho_{AB}^{1/2}) = \sigma_{AB}^{-1/2}\rho_{AB}^{1/2}\rho_{AB}^{1/2} = \sigma_{AB}^{-1/2}\rho_{AB}$

$\Delta_{AB} = \sigma_{AB}\cdot\rho_{AB}^{-1}$

$\mathcal{U}(X_A) = (X_A \rho_A^{-1/2}\otimes \mathbb{1}_B)\rho_{AB}^{1/2}$

RHS: $\mathcal{U}\,\Delta_A^{-1/2}(\rho_A^{1/2}) = \mathcal{U}(\sigma_A^{-1/2}\rho_A)$

$$= (\sigma_A^{-1/2}\rho_A\rho_A^{-1/2}\otimes \mathbb{1}_B)\rho_{AB}^{1/2}$$

$$= (\sigma_A^{-1/2}\rho_A^{1/2}\otimes \mathbb{1}_B)\rho_{AB}^{1/2}$$

LHS = RHS $\Leftrightarrow$ $\sigma_{AB}^{-1/2}\rho_{AB} = (\sigma_A^{-1/2}\rho_A^{1/2}\otimes\mathbb{1}_B)\rho_{AB}^{1/2}$

$$\rho_{AB}^{1/2} = \sigma_{AB}^{1/2}(\sigma_A^{-1/2}\rho_A^{1/2}\otimes \mathbb{1}_B)$$

$$\rho_{AB} = \rho_{AB}^{1/2}(\rho_{AB}^{1/2})^\dagger = \sigma_{AB}^{1/2}(\sigma_A^{-1/2}\rho_A\sigma_A^{-1/2}\otimes\mathbb{1}_B)\sigma_{AB}^{1/2} \qquad \square$$

$$\left(R_\sigma(X_A) = \sigma_{AB}^{1/2}(\sigma_A^{-1/2}X_A\sigma_A^{-1/2}\otimes\mathbb{1}_B)\sigma_{AB}^{1/2}\right)$$