

## MATH 595 Quantum channels I: Representations and properties.

Exercise sheet 2 – July 22, 2021

Unless stated otherwise,  $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \dots$  denote finite-dimensional Hilbert spaces.

1. Prove the following statements:

(a)  $\frac{1}{2}\mathbb{1} = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$  for any  $\rho \in \mathcal{B}(\mathbb{C}^2)$ .

*Hint: You can show this by brute-force computation; a more elegant proof is based on Schur's Lemma.*

(b) Cyclicity property of the partial trace: For any  $X \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  and  $Y \in \mathcal{B}(\mathcal{H}_2)$ ,

$$\text{tr}_2 X(\mathbb{1} \otimes Y) = \text{tr}_2(\mathbb{1} \otimes Y)X. \quad (1)$$

(c) The set of antidegradable channels is convex.

(d) Let  $\vartheta: X \mapsto X^T$ . The map  $\vartheta \circ T \circ \vartheta$  is CP iff  $T$  is CP.

2. Any linear operator  $W \in \mathcal{B}(\mathbb{C}^2)$  can be written as

$$W = \frac{1}{2}(w_0\mathbb{1} + w_1X + w_2Y + w_3Z) \quad (2)$$

with  $w_i \in \mathbb{C}$ . Show the following:

(a)  $W$  is Hermitian iff  $w_i \in \mathbb{R}$ .

(b)  $W$  has unit trace iff  $w_0 = 1$ .

(c)  $W$  is positive semidefinite iff  $\|\mathbf{w}\|_2 \leq w_0$ , where  $\mathbf{w} = (w_x, w_y, w_z)^T \in \mathbb{R}^3$  is called the *Bloch vector*.

(d)  $W$  is a pure qubit state iff  $\|\mathbf{w}\|_2 = 1$ .

3. Setting  $\sigma = (X, Y, Z)$ , a qubit-qubit channel  $\mathcal{N}: \mathcal{B}(\mathbb{C}^2) \rightarrow \mathcal{B}(\mathbb{C}^2)$  can be written with respect to the basis (2) as

$$\mathcal{N} \left( \frac{1}{2}(w_0\mathbb{1} + w_1X + w_2Y + w_3Z) \right) = \frac{1}{2} (w_0\mathbb{1} + (\mathbf{t} + T\mathbf{w}) \cdot \sigma), \quad (3)$$

where  $\mathbf{t} \in \mathbb{R}^3$  and  $T$  is a real  $3 \times 3$ -matrix. Alternatively, one may write the action of  $\mathcal{N}$  on the vector  $(w_0, w_1, w_2, w_3)^T$  in terms of a  $4 \times 4$ -matrix

$$N = \begin{pmatrix} 1 & 0 \\ \mathbf{t} & T \end{pmatrix}. \quad (4)$$

The fact that  $N_{1j} = 0$  for  $j = 2, 3, 4$  reflects the trace-preserving condition for  $\mathcal{N}$ . Show the following:

- (a)  $\mathcal{N}$  is unital iff  $\mathbf{t} = \mathbf{0}$ .
  - (b)  $\mathcal{N}$  is a Pauli channel,  $\mathcal{N}: \rho \mapsto p_0\rho + p_1X\rho X + p_2Y\rho Y + p_3Z\rho Z$ , iff  $\mathbf{t} = \mathbf{0}$  and the corresponding matrix  $T$  defined via (3) is diagonal.<sup>1</sup>
  - (c) Use 3b to show that any unital qubit channel  $\mathcal{N}$  is unitarily equivalent to a Pauli channel, i.e., there exist unitaries  $U$  and  $V$  such that  $U\mathcal{N}(V \cdot V^\dagger)U^\dagger$  is a Pauli channel.  
*Hint: Use the singular value decomposition for the matrix  $T$ . Linear transformations acting on  $\mathbf{w} \in \mathbb{R}^3$  are mapped to linear transformations on  $\mathbb{C}^2$  via (3). What is the image of an orthogonal transformation under this mapping?*  
*Comment: Following Footnote 1, the resulting Pauli map is indeed CP if  $\mathcal{N}$  is a unital channel.*
  - (d) Use 3c to show that every unital qubit channel  $\mathcal{N}$  is a mixed-unitary channel, i.e., there are unitaries  $U_1, \dots, U_k$  and a probability distribution  $(p_1, \dots, p_d)$  such that  $\mathcal{N} = \sum_{i=1}^k p_i U_i \cdot U_i^\dagger$ .
4. Let  $\mathcal{A}_\gamma$  with  $\gamma \in [0, 1]$  be the amplitude damping channel with Kraus operators  $K_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$  and  $K_1 = \sqrt{\gamma}|0\rangle\langle 1|$ .
- (a) Determine the limit  $\lim_{n \rightarrow \infty} \mathcal{A}_\gamma^n(\rho)$  for an arbitrary input density matrix  $\rho$ , where  $\mathcal{A}_\gamma^n := \mathcal{A}_\gamma \circ \dots \circ \mathcal{A}_\gamma$  denotes the  $n$ -fold composition of the channel with itself. What state is the completely mixed state  $\frac{1}{2}\mathbb{1}$  mapped to in this process? What can you say about the entropies of the initial and the final state in this case?
  - (b) Show that  $\mathcal{A}_\gamma$  is degradable for  $\gamma \in [0, 1/2]$  and antidegradable for  $\gamma \in [1/2, 1]$ .

5. Determine complementary channels of:

- (a) the erasure channel  $\mathcal{E}_p: \rho \mapsto (1-p)\rho + p \operatorname{tr}(\rho)|e\rangle\langle e|$ ;
- (b) the  $Z$ -dephasing channel  $\mathcal{F}_p^Z: \rho \mapsto (1-p)\rho + Z\rho Z$ .

6. Recall that a generalized dephasing channel on  $\mathbb{C}^d$  is defined in the following way: Let  $\{|i\rangle\}_{i=1}^d$  be an arbitrary orthonormal basis for  $\mathbb{C}^d$ , and let  $\{|\phi_i\rangle_E\}_{i=1}^d$  be an arbitrary set of

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<sup>1</sup> Let  $T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}$ , then the corresponding Pauli channel has parameters

$$p_0 = \frac{1}{4}(t_1 + t_2 + t_3 + 1) \quad p_1 = \frac{1}{4}(t_1 - t_2 - t_3 + 1) \quad p_2 = \frac{1}{4}(-t_1 + t_2 - t_3 + 1) \quad p_3 = \frac{1}{4}(-t_1 - t_2 + t_3 + 1), \quad (5)$$

which defines a probability distribution if and only if

$$1 \pm t_3 \geq |t_1 \pm t_2|. \quad (6)$$

These relations coincide with the complete positivity conditions for unital maps defined via (3), as proved by King & Ruskai in [arXiv:quant-ph/9911079](https://arxiv.org/abs/quant-ph/9911079).

vectors in an auxiliary Hilbert space  $\mathcal{H}_E$  with  $\dim \mathcal{H}_E \geq 2$ . Then the channel isometry is defined by the action on this basis as  $|i\rangle \mapsto |i\rangle \otimes |\phi_i\rangle_E$ .

- (a) Determine a basis  $\{|i\rangle\}_{i=0,1}$  and vectors  $\{|\phi_i\rangle_E\}_{i=0,1}$  for the Z-dephasing channel in 5b.
- (b) Prove that any complementary channel of a generalized dephasing channel is entanglement-breaking.
- (c) Prove that generalized dephasing channels are degradable.  
*Hint: Either explicitly construct a degrading map  $\mathcal{D}$  satisfying  $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$  where  $\mathcal{N}$  is a generalized dephasing channel, or use 6b directly.*
- (d) Let  $U_1, \dots, U_k \in \mathcal{U}(\mathbb{C}^d)$  be unitaries,  $(p_1, \dots, p_k)$  be a probability distribution, and  $\mathcal{N} = \sum_{i=1}^k p_i U_i \cdot U_i^\dagger$  be a mixed unitary channel. Give a sufficient condition for  $\mathcal{N}$  to be a generalized dephasing channel.