Recap

- channel twirling: averaging operation on channels that enforces covariance property
- Let $\mathcal{N}: A \to B$ be a channel, G a finite group with unitary rep's U_g on $\mathcal{H}_A$, V_g on $\mathcal{H}_B$: $\mathcal{N}_G(\cdot) = \frac{1}{|G|} \sum_{g \in G} V_g^\dagger \mathcal{N}(U_g \cdot U_g^\dagger) V_g$
- $\mathcal{N}_G$ is (G, U_g, V_g)-covariant.
- For continuous groups: need suitable averaging operation
- Relevant case for us: compact Lie groups (such as unitary group)

  For such groups there exists a unique "uniform" probability measure $\mu: \sigma_G \to \mathbb{R}^+$ satisfying left- and right-invariance:

$$\mu(S) = \mu(gS) = \mu(Sg) \quad \forall g \in G,\ S \in \sigma_G$$

- This measure is called Haar measure and gives rise to the Haar integral $\int_G d\mu(g)\, f(g)$
- For finite groups, μ is the counting measure and

$$\int_G d\mu(g)\, f(g) = \frac{1}{|G|} \sum_{g \in G} f(g)$$

- Channel twirl: $\mathcal{N}_G = \int_G d\mu(g)\, V_g^\dagger \mathcal{N}(U_g \cdot U_g^\dagger) V_g$
- Result: $\mathcal{N}_{\mathcal{U}(d)} = \int_{\mathcal{U}(d)} dU\, U^\dagger \mathcal{N}(U \cdot U^\dagger) U$ is a depolarizing channel.

Lemma 22 An irrep (φ, R) of a group G (finite or compact) forms a so-called 1-design: $\frac{1}{|G|} \sum_{g \in G} \varphi(g) X \varphi(g)^{\dagger} = \frac{\operatorname{tr} X}{d} \mathbb{1}_R$,

where $X \in B(R)$, and $d = \dim R$.

Proof: Schur's lemma. $\square$

Prop 23 Let $N: B(\mathcal{H}) \to B(K)$ be a (G, U_g, V_g)-covariant channel where $g \mapsto V_g$ on K is irreducible.

Then N satisfies $N\left(\frac{1}{|\mathcal{H}|} \mathbb{1}_{\mathcal{H}}\right) = \frac{1}{|K|} \mathbb{1}_K$.

If furthermore $|\mathcal{H}| = |K|$ (i.e., $\mathcal{H} \cong K$), then N is unital.

Proof: $V_g N(\cdot) V_g^{\dagger} = N(U_g \cdot U_g^{\dagger}) \quad \forall g \in G$

$$N(\mathbb{1}_{\mathcal{H}}) = N(U_g \mathbb{1} U_g^{\dagger}) = V_g N(\mathbb{1}) V_g^{\dagger} \qquad \forall g \in G$$

Lemma 22 $\Rightarrow$ (Schur's lemma)

$$N(\mathbb{1}_{\mathcal{H}}) = \frac{\operatorname{tr} N(\mathbb{1})}{|K|} \mathbb{1}_K = \frac{|\mathcal{H}|}{|K|} \mathbb{1}_K$$

$\operatorname{tr} N(\mathbb{1}_{\mathcal{H}}) = \operatorname{tr} \mathbb{1}_{\mathcal{H}}$

$$\Rightarrow N\left(\frac{1}{|\mathcal{H}|} \mathbb{1}_{\mathcal{H}}\right) = \frac{1}{|K|} \mathbb{1}_K, \text{ and } N(\mathbb{1}_{\mathcal{H}}) = \mathbb{1}_K \text{ if } |\mathcal{H}| = |K|. \quad \square$$

For irreducibly covariant channels, the Holevo information assumes a simple form.

Reminder: in general,

$$\chi(\mathcal{N}) = \max_{\{p_x, \rho_x\}} \left\{ S\left(\mathcal{N}\left(\sum_x p_x \rho_x\right)\right) - \sum_x p_x \, S(\mathcal{N}(\rho_x)) \right\}$$

Lemma 24 The maximization in $\chi(\mathcal{N})$ can be restricted to pure state ensembles.

Proof: Let $\{p_x, \rho_x\}$ be a (mixed) state ensemble achieving $\chi(\mathcal{N})$:

$$\chi(\mathcal{N}) = S\left(\sum_x p_x \mathcal{N}(\underbrace{\rho_x}_{=\rho_A^x})\right) - \sum_x p_x \, S(\mathcal{N}(\rho_x))$$

Let $\rho_{XA} = \sum_x p_x \, |x\rangle\langle x|_X \otimes \rho_A^x$ (so-called classical-quantum state)

$$= \begin{pmatrix} p_1 \rho_A^1 & & \\ 0 & p_2 \rho_A^2 & \\ & & \ddots \end{pmatrix} = \bigoplus_x p_x \rho_A^x$$

$$\sigma_{XB} = (\mathrm{id}_X \otimes \mathcal{N})(\rho_{XA}) = \sum_x p_x \, |x\rangle\langle x|_X \otimes \mathcal{N}(\rho_A^x)$$

$$S\left(\mathcal{N}\left(\sum_x p_x \rho_A^x\right)\right) = S(\sigma_B) \qquad \underbrace{-\sum_x p_x \log p_x}_{=S(\sigma_X)} = H(\{p_x\})$$

$$\sum_x p_x \, S(\mathcal{N}(\rho_A^x)) = S(\sigma_{XB}) - S(\sigma_X)$$

$$S\Big(\mathcal{N}\Big(\sum_x p_x \rho_A^x\Big)\Big) = S(\sigma_B)$$

$$-\sum_x p_x \log p_x = H(\{p_x\}) \quad \text{Shannon entropy}$$

$$\sum_x p_x \, S(\mathcal{N}(\rho_A^x)) = S(\sigma_{XB}) - S(\sigma_X)$$

$$\chi(\mathcal{N}) = S(\sigma_B) - \big(S(\sigma_{XB}) - S(\sigma_X)\big)$$

$$= S(\sigma_B) + S(\sigma_X) - S(\sigma_{XB})$$

$$= I(X;B)_\sigma \quad \text{mutual information (between } X \text{ and } B)$$

$$\forall x: \ \rho_A^x = \sum_y p_{y|x} |\psi_{x,y}\rangle\langle\psi_{x,y}|$$

$$\sigma_{XYB} = \sum_{x,y} \underbrace{p_x p_{y|x}}_{p_{x,y}} |x\rangle\langle x|_X \otimes |y\rangle\langle y|_Y \otimes \mathcal{N}(|\psi_{x,y}\rangle\langle\psi_{x,y}|)$$

$$\sigma_{XB} = \mathrm{tr}_Y \, \sigma_{XYB}$$

$$I(XY;B)_\sigma \geq I(X;B)_\sigma = \underline{\chi(\mathcal{N})} = I(Y;B)$$

data-processing inequality (will be proved and discussed in QC2)

$\square$

$\boxed{\text{Prop 25}}$ Let $\mathcal{N}: B(\mathcal{H}) \to B(\mathcal{K})$ be a (G, U_g, V_g)-covariant channel.

i) If $g \mapsto U_g$ on $\mathcal{H}$ is irreducible, then $(d_1 = \dim \mathcal{H})$

$$\chi(\mathcal{N}) = S\left(\mathcal{N}\left(\tfrac{1}{d}\mathbb{1}_{\mathcal{H}}\right)\right) - \min_{|\psi\rangle} S(\mathcal{N}(\psi)).$$

ii) if in addition $g \mapsto V_g$ is irred., then $\mathcal{N}\left(\frac{1}{d_1}\mathbb{1}_{\mathcal{H}}\right) = \frac{1}{d_2}\mathbb{1}_{\mathcal{K}}$ $(d_2 = |\mathcal{K}|)$

$$\chi(\mathcal{N}) = \log d_2 - \min_{|\psi\rangle} S(\mathcal{N}(\psi)).$$

<u>Proof:</u> i) Prop 24: Let $\{p_x, |\psi_x\rangle\langle\psi_x|\}$ achieve max. in $\chi(\mathcal{N})$:

$$\chi(\mathcal{N}) = S\left(\sum_x p_x \mathcal{N}(\psi_x)\right) - \sum_x p_x S(\mathcal{N}(\psi_x))$$

we have the following bounds:

$\cdot)$ $\sum_x p_x \underbrace{S(\mathcal{N}(\psi_x))}_{\geq \min_{|\psi\rangle} S(\mathcal{N}(\psi))} \geq \min_{|\psi\rangle} S(\mathcal{N}(\psi))$

$S(\rho) = S(U\rho U^+)$ $\forall \rho \geq 0$, U unitary

$\cdot)$ $\forall g$ $S\left(\sum_x p_x \mathcal{N}(\psi_x)\right) = S\left(\sum_x p_x V_g \mathcal{N}(\psi_x) V_g^+\right)$

$$S\Big(\sum_x p_x \mathcal{N}(\psi_x)\Big) = \frac{1}{|G|}\sum_{g\in G} S\Big(\sum_x p_x \underbrace{V_g \mathcal{N}(\psi_x) V_g^\dagger}_{\mathcal{N}(U_g \psi_x U_g^\dagger)}\Big)$$

$$= \frac{1}{|G|}\sum_{g\in G} S\Big(\sum_x p_x \mathcal{N}(U_g \psi_x U_g^\dagger)\Big)$$

van Neumann entropy is concave (Ex. sheet 3) $\rightarrow$

$$\leq S\Big(\sum_x p_x \mathcal{N}\Big(\underbrace{\frac{1}{|G|}\sum_{g\in G} U_g \psi_x U_g^\dagger}_{=\frac{1}{d}\mathbb{1}_X \ \forall x \text{ by lemma 22}}\Big)\Big)$$

$$= S\Big(\mathcal{N}\Big(\frac{1}{d}\mathbb{1}_X\Big)\Big)$$

In summary, $\chi(\mathcal{N}) = S\Big(\sum_x p_x \mathcal{N}(\psi_x)\Big) - \sum_x p_x S(\mathcal{N}(\psi_x))$

$$\leq S\Big(\mathcal{N}\Big(\frac{1}{d}\mathbb{1}_X\Big)\Big) - \min_{|\psi\rangle} S(\mathcal{N}(\psi))$$

This value is achieved by the following pure-state ensemble:

Let $|\varphi\rangle$ achieve the minimum in $\min_{|\psi\rangle} S(\mathcal{N}(\psi))$

Define $|\psi_g\rangle = U_g|\varphi\rangle$ and $p_g = \frac{1}{|G|}$

$$\chi(\{p_g, |\psi_g\rangle\}, \mathcal{N}) = S\Big(\underbrace{\sum_g \frac{1}{|G|}\mathcal{N}(U_g \varphi U_g^\dagger)}_{\mathcal{N}(\frac{1}{d}\mathbb{1})}\Big) - \frac{1}{|G|}\sum_{g\in G} S\big(\overbrace{\mathcal{N}(U_g\varphi U_g^\dagger)}^{S(V_g\mathcal{N}(\varphi)V_g^\dagger) \,=\, S(\mathcal{N}(\varphi))}\big)$$

ii) If $g \mapsto V_g$ is irreducible as well, then by Prop 23,

$$\mathcal{N}\left(\frac{1}{d_1}\mathbb{1}_{\mathcal{X}}\right) = \frac{1}{d_2}\mathbb{1}_{\mathcal{K}}, \qquad (d_1 = |\mathcal{X}|,\ d_2 = |\mathcal{K}|)$$

so that $S\left(\mathcal{N}\left(\frac{1}{d}\mathbb{1}_{\mathcal{X}}\right)\right) = S\left(\frac{1}{d_2}\mathbb{1}_{\mathcal{K}}\right) = \log d_2$. $\square$