Recap

·) A channel $\mathcal{N}: A \to B$ with complementary channel $\mathcal{N}^c: A \to E$ is called antidegradable, if there is an antidegrading map $\mathcal{A}: E \to B$ s.t.

$$\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$$

·) Antidegradable channels have zero quantum capacity (no-cloning).

·) $\mathcal{N}$ is antidegradable $\iff \tau^{\mathcal{N}}_{AB}$ has a symmetric extension:

$$\exists\, \sigma_{ABB'} \text{ s.t. } \mathrm{tr}_B\, \sigma = \mathrm{tr}_{B'}\, \sigma = \tau^{\mathcal{N}}$$

$$\mathbb{F}_{BB'}\, \sigma\, \mathbb{F}_{BB'} = \sigma .$$

·) Any EB channel is antidegradable

$\Leftarrow$ separable states are k-extendible $\forall k$.

·) Examples of antidegradable channels:

- erasure channel $\mathcal{E}_p: \rho \mapsto (1-p)\rho + p\,\mathrm{tr}(\rho)\, |e\rangle\langle e|$ for $p \geq \frac{1}{2}$.
- amplitude damping channel $\mathcal{A}_\gamma$: $|0\rangle \mapsto |00\rangle$
  $|1\rangle \mapsto \sqrt{1-\gamma}\, |10\rangle + \sqrt{\gamma}\, |01\rangle$

  for $\gamma \geq \frac{1}{2}$.
- depolarizing channel $\mathcal{D}_p: \rho \mapsto (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$

  for $p \geq \frac{1}{4}$.

§2.6. Degradable channels

Dual concept to antidegradable channels:

$$\begin{tikzcd} & B \arrow[dd, "\mathcal{D}"] \\ A \arrow[ur, "\mathcal{N}"] \arrow[dr, "\mathcal{N}^c"'] & \\ & E \end{tikzcd}$$

$\boxed{\text{Def 15}}$ A channel $\mathcal{N}: A \to B$ with complementary channel $\mathcal{N}^c: A \to E$ is called degradable, if $\exists$ channel $\mathcal{D}: B \to E$ (degrading map) s.t. $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$.

$\mathcal{N}$ is degradable iff $\mathcal{N}^c$ is antidegradable

Intuition: Bob can locally 'simulate' the environment

$\Rightarrow$ we understand the quantum capacity of degradable channels (and efficiently computable as well!)

·) there are degradable channels $\mathcal{N}$ s.t. $Q(\mathcal{N}) > 0$.

Examples:

(we have already proved this!)

·) erasure channel $\mathcal{E}_p$ for $p \leq \frac{1}{2}$ $\quad (Q(\mathcal{E}_p) = 1 - 2p)$

·) amplitude damping channel $\mathcal{A}_\gamma$ for $\gamma \leq \frac{1}{2}$.

·) generalized dephasing channels

·) the complementary channel of any entanglement-breaking channel
"Hadamard channels"

$\boxed{\text{No-cloning thm}}$ there is no linear op. $|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle \ \forall |\psi\rangle$

Fundamental differences between DEG / ADG:

·) $\mathcal{N}$ DEG: $Q(\mathcal{N}) \geq 0$ / $\mathcal{N}$ ADG: $Q(\mathcal{N}) = 0$

·) DEG is not convex / $\mathcal{N}_1, \mathcal{N}_2$ ADG: $\lambda \mathcal{N}_1 + (1-\lambda)\mathcal{N}_2$ ADG

$\to$ ADG is a convex set (Ex)

$$\Big\downarrow \text{"reason"}$$

$\mathcal{N}_1$ DEG with Choi op ρ_{AB}
$\mathcal{N}_2$ DEG with Choi op σ_{AB}

$\underline{\lambda \mathcal{N}_1 + (1-\lambda)\mathcal{N}_2} = \mathcal{N}$ with Choi op. ω_{AB}

$|\psi^\rho\rangle_{ABE}$ purifies ρ_{AB}, $|\psi^\sigma\rangle_{ABE}$ purifies σ_{AB}

$|\gamma^\omega\rangle_{ABEX} = \sqrt{\lambda}\, |\psi^\rho\rangle_{ABE} |0\rangle_X + \sqrt{1-\lambda}\, |\psi^\sigma\rangle_{ABE} |1\rangle_X$ purifies ω_{AB}: $tr_{EX}\, \gamma^\omega = \omega_{AB}$

Choi op of $\mathcal{N}^c$: $\omega_{AEX} = \begin{pmatrix} \lambda \rho_{AE} & \sqrt{\lambda(1-\lambda)}\, tr_B |\psi^\rho\rangle\langle\psi^\sigma|_{ABE} \\ h.c. & (1-\lambda)\sigma_{AE} \end{pmatrix}$

$\boxed{\text{Prop}}$ (without proof)

A qubit-qubit channel with qubit environment $(|A| = |B| = |E| = 2)$ is degradable or antidegradable.

Proof idea: $\mathcal{N}$ DEG: $\exists\, \mathcal{D}: B \to E$ s.t. $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$

using transfer matrices: $N^c = D \cdot N \overset{N \text{ inv.}}{\iff} \underline{D = N^c \cdot N^{-1}}$

degradability $\iff$ CP of $\mathcal{D}$ defined via

inv.

$\mathcal{N}$ ADG: $\exists\, \mathcal{A}: E \to B$ s.t. $\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$

using transfer matrices: $N = A \cdot N^c \overset{\mathcal{N}, \mathcal{N}^c \text{ inv.}}{\iff} \underline{A = N \cdot (N^c)^{-1}}$

note: $A = D^{-1}$

anti-degradability $\iff$ CP of $\mathcal{A}$ defined via

For $|A| = |B| = |E| = 2$, we always have that D or D^{-1}

defines a completely positive map. $\xrightarrow{\cdots}$ "$\square$"

(proof can be found in arXiv: quant-ph/0607070)

Section 3: Covariant channels and minimum output entropy

§3.1 Definition

Motivation: $\mathcal{N}: A \to B$, U, V unitaries:

$$\mathcal{M}(X) = U\, \mathcal{N}(V X V^\dagger)\, U^\dagger$$

is unitarily equivalent to $\mathcal{N}$.

From an info-th. point of view, $\mathcal{N}$ and $\mathcal{M}$ are equivalent in that they have the same capacities:

Any protocol for $\mathcal{N}$ (achieving e.g. entanglement generation) can be turned into a protocol for $\mathcal{M}$ achieving the same task with the same rate by absorbing U and V into the protocol.

For certain channels $\mathcal{N}$ and unitaries U, V we have $\mathcal{M} = \mathcal{N}$, i.e., (U, V) is a symmetry of the channel $\to$ covariance.

Basics from representation theory

G is a group (for us: G finite or compact)

·) A representation of G on a vector space R is a group homomorphism $\varphi: G \to GL(R)$, i.e., $\varphi(gh) = \varphi(g)\cdot\varphi(h)$ for $g, h \in G$

(φ, R) is finite-dimensional if $\dim_{\mathbb{C}} R < \infty$.

$(\varphi, \mathcal{R})$ is called a unitary representation, if $\mathcal{R}$ is a Hilbert space and $\varphi(g)$ is unitary $\forall g \in G$. $\quad (\varphi : G \to \mathcal{U}(\mathcal{R}))$

·) a subspace $S \leq \mathcal{R}$ is called G-invariant, if $\varphi(g)s \in S$ $\forall g \in G$ $\forall s \in S$

$\{0\}, \mathcal{R}$ are always G-invariant subspaces of $(\varphi, \mathcal{R})$

·) A representation $(\varphi, \mathcal{R})$ is called irreducible, if $\{0\}$ and $\mathcal{R}$ are the only G-invariant subspaces. (irrep $\equiv$ irreducible rep.)

·) Finite-dimensional unitary representations of any group are completely reducible: $\varphi = \bigoplus_i \varphi_i$ where φ_i are irreducible.

·) Let $(\varphi_V, V), (\varphi_W, W)$ be rep's of a group G.

A linear map $f : V \to W$ is called G-linear, if $f \circ \varphi_V(g) = \varphi_W(g) \circ f$ $\forall g \in G$.

Schur's lemma Let φ_V, φ_W be irreps of a group G, and assume $f : V \to W$ is G-linear.

Then either: ·) $V \ncong W$ and $f = 0$

or: ·) $V \cong W$ and $f = \lambda \, id_{V \to W}$ for some $\lambda \in \mathbb{C}$

$\boxed{\text{Def 16}}$ Let $\mathcal{N}: A \to B$ be a quantum channel, and G be a group with unitary representations U_g on $\mathcal{H}_A$ and V_g on $\mathcal{H}_B$. Then $\mathcal{N}$ is called covariant w.r.t. (G, U_g, V_g) if

$$V_g \mathcal{N}(.) V_g^\dagger = \mathcal{N}(U_g . U_g^\dagger) \quad \forall g \in G.$$