

Recap

.) A channel $\mathcal{N}: A \rightarrow B$ with complementary channel $\mathcal{N}^c: A \rightarrow E$ is called **antidegradable**, if there is an antidegrading map $A: E \rightarrow B$ s.t.

$$\mathcal{N} = A \circ \mathcal{N}^c$$

.) Antidegradable channels have **zero quantum capacity (no-cloning)**.

.) \mathcal{N} is antidegradable $\Leftrightarrow \tau_{AB}^{\mathcal{N}}$ has a symmetric extension:

$$\exists \sigma_{ABE'} \text{ s.t. } \text{tr}_B \sigma = \text{tr}_{B'} \sigma = \tau_{AB}^{\mathcal{N}}$$

$$\mathbb{F}_{BB'} \subseteq \mathbb{F}_{B'B} = \sigma.$$

.) Any EB channel is antidegradable

\Leftrightarrow separable states are k -extendible $\forall k$.

.) Examples of antidegradable channels:

- **erase channel** $\mathcal{E}_p: \rho \mapsto (1-p)\rho + p \text{tr}(\rho) |e\rangle\langle e|$ for $p \geq \frac{1}{2}$.

- **amplitude damping channel** $\mathcal{A}_\gamma: |0\rangle \mapsto |00\rangle$
 $|1\rangle \mapsto \sqrt{1-\gamma} |10\rangle + \sqrt{\gamma} |01\rangle$

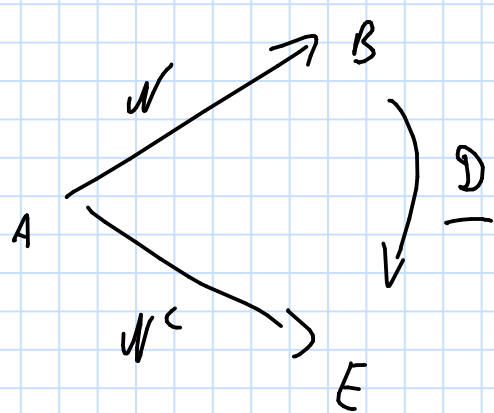
for $\gamma \geq \frac{1}{2}$.

- **depolarizing channel** $\mathcal{D}_p: \rho \mapsto (1-p)\rho + \frac{p}{3} (X\rho X + Y\rho Y + Z\rho Z)$

for $p \geq \frac{1}{4}$.

§ 2.6. Degradable channels

Dual concept to antidegradable channels:



Def 15 A channel $N: A \rightarrow B$ with complementary channel $N^c: A \rightarrow E$ is called degradable, if \exists channel $D: B \rightarrow E$ (degrading map) s.t. $N^c = D \circ N$.

N is degradable iff N^c is antidegradable

Intuition: Bob can locally "simulate" the environment

\Rightarrow we understand the quantum capacity of degradable channels (and efficiently computable as well!)

\rightarrow there are degradable channels N s.t. $Q(N) > 0$.

Examples:

\rightarrow erasure channel \mathcal{E}_p for $p \leq \frac{1}{2}$ (we have already proved this!)
($Q(\mathcal{E}_p) = 1 - 2p$)

\rightarrow amplitude damping channel A_γ for $\gamma \leq \frac{1}{2}$.

\rightarrow generalized dephasing channels

\rightarrow the complementary channel of any entanglement-breaking channel
"Hadamard channels"

No-cloning theorem

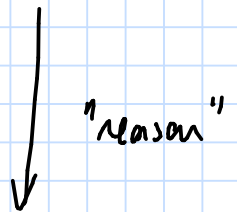
there is no linear op. $|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle \quad \forall |\psi\rangle$

Fundamental differences between DEG / ADG:

1) N DEG: $Q(N) \geq 0$ / N ADG: $Q(N) = 0$

2) DEG is not convex / N_1, N_2 ADG: $\lambda N_1 + (1-\lambda) N_2$ ADG

\rightarrow ADG is a convex set (Ex)



$$\left. \begin{array}{l} N_1 \text{ DEG with Choi op } \rho_{AB} \\ N_2 \text{ DEG with Choi op } \sigma_{AB} \end{array} \right\} \underline{\lambda N_1 + (1-\lambda) N_2 = N \text{ with Choi op. } \omega_{AB}}$$

$|\psi^s\rangle_{ABE}$ purifies ρ_{AB} , $|\psi^s\rangle_{ABE}$ purifies σ_{AB}

$|\chi^w\rangle_{ABEX} = \sqrt{\lambda} |\psi^s\rangle_{ABE} |0\rangle_X + \sqrt{1-\lambda} |\psi^s\rangle_{ABE} |1\rangle_X$ purifies ω_{AB} : $\text{tr}_{EX} \chi^w = \omega_{AB}$

Choi op of N^c :
$$W_{AEX} = \begin{pmatrix} \lambda \rho_{AE} & \sqrt{\lambda(1-\lambda)} \text{tr}_B |\psi^s \rangle \langle \psi^s|_{ABE} \\ \text{h.c.} & (1-\lambda) \sigma_{AE} \end{pmatrix}$$

Prop (without proof)

A qubit-qubit channel with qubit environment ($|A| = |B| = |E| = 2$) is degradable or antidegradable.

Proof idea: $N \in \mathcal{L}; \exists \mathcal{D}: B \rightarrow E$ s.t. $N^c = \mathcal{D} \circ N$

using transfer matrices: $N^c = \mathcal{D} \cdot N \stackrel{N \text{ inv.}}{\Leftrightarrow} \underline{\mathcal{D} = N^c \cdot N^{-1}}$

degradability \Leftrightarrow CP of \mathcal{D} defined via

inv. $\left. \begin{array}{l} \text{inv.} \\ \text{inv.} \end{array} \right\}$

$N \in \mathcal{L}; \exists A: E \rightarrow B$ s.t. $N = A \cdot N^c$

using transfer matrices: $N = A \cdot N^c \stackrel{N, N^c \text{ inv.}}{\Leftrightarrow} \underline{A = N \cdot (N^c)^{-1}}$

note: $A = \mathcal{D}^{-1}$

anti-degradability \Leftrightarrow CP of A defined via

For $|A| = |B| = |E| = 2$, we always have that \mathcal{D} or \mathcal{D}^{-1}

defines a completely positive map. $\dots \rightarrow \square$

(proof can be found in arXiv: quant-ph/0607070)

Section 3: Covariant channels and minimum entropy

§ 3.1 Definition

Motivation: $N: A \rightarrow B$, U, V unitaries:

$$M(X) = U N(V X V^\dagger) U^\dagger$$

is unitarily equivalent to N .

From an info-th. point of view, N and M are equivalent in that

they have the same capacities:

Any protocol for N (achieving e.g. entanglement generation)

can be turned into a protocol for M achieving the same task

with the same rate by absorbing U and V into the protocol.

For certain channels N and unitaries U, V we have $M = N$,

i.e., (U, V) is a symmetry of the channel \rightarrow covariance.

Basics from representation theory

G is a group (for us: G finite or compact)

.) A representation of G on a vector space \mathbb{R} is a group homomorphism

$$\varphi: G \rightarrow GL(\mathbb{R}), \text{ i.e., } \varphi(gh) = \varphi(g) \cdot \varphi(h) \text{ for } g, h \in G$$

(φ, \mathbb{R}) is finite-dimensional if $\dim_{\mathbb{R}} \mathbb{R} < \infty$.

(φ, \mathcal{R}) is called a unitary representation, if \mathcal{R} is a Hilbert space and $\varphi(g)$ is unitary $\forall g \in G$. $(\varphi: G \rightarrow \mathcal{U}(\mathcal{R}))$

.) a subspace $S \subseteq \mathcal{R}$ is called G -invariant, if $\varphi(g)s \in S \forall g \in G$
 $\forall s \in S$

$\{0\}, \mathcal{R}$ are always G -invariant subspaces of (φ, \mathcal{R})

.) A representation (φ, \mathcal{R}) is called irreducible, if $\{0\}$ and \mathcal{R} are the only G -invariant subspaces. (irrep \equiv irreducible rep.)

.) Finite-dimensional unitary representations of any group are completely reducible: $\varphi = \bigoplus_i \varphi_i$ where φ_i are irreducible.

.) Let $(\varphi_V, V), (\varphi_W, W)$ be reps of a group G .

A linear map $f: V \rightarrow W$ is called G -linear, if $f \circ \varphi_V(g) = \varphi_W(g) \circ f \forall g \in G$.

Schur's lemma Let φ_V, φ_W be irreps of a group G , and

assume $f: V \rightarrow W$ is G -linear.

Then either: .) $V \not\cong W$ and $f = 0$

or: .) $V \cong W$ and $f = \lambda \text{id}_{V \rightarrow W}$ for some $\lambda \in \mathbb{C}$

Def 16 Let $\mathcal{N}: A \rightarrow B$ be a quantum channel, and G be a group with unitary representations U_g on \mathcal{H}_A and V_g on \mathcal{H}_B .

Then \mathcal{N} is called covariant w.r.t. (G, U_g, V_g) if

$$V_g \mathcal{N}(\cdot) V_g^\dagger = \mathcal{N}(U_g \cdot U_g^\dagger) \quad \forall g \in G.$$