Recap

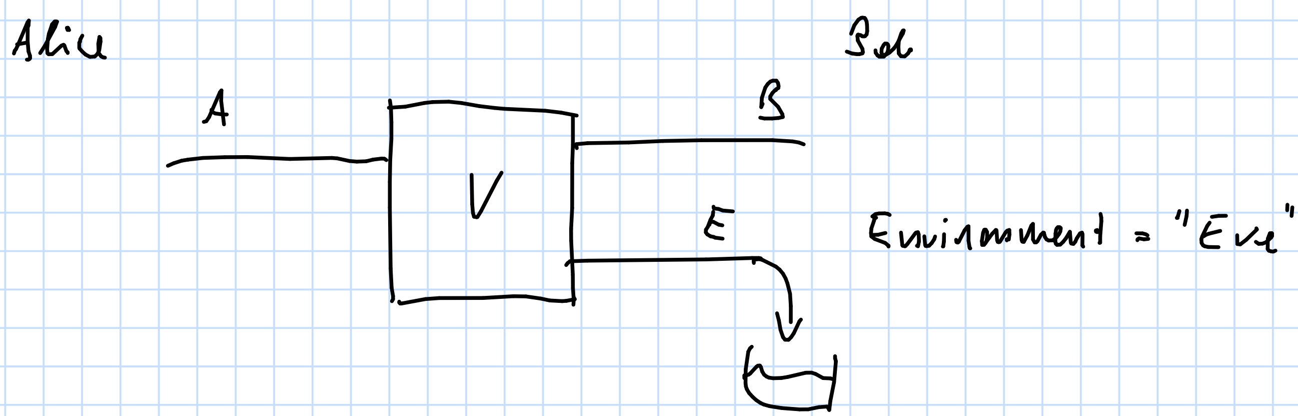
- Linear representation of quantum channels:
  - view $\mathcal{B}(\mathcal{H})$ as Hilbert space with Hilbert-Schmidt inner prod
  $$\langle X, Y\rangle = \mathrm{tr}(X^\dagger Y)$$
  - represent linear maps $T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ as matrices acting on "vectors" $X \in \mathcal{B}(\mathcal{H})$.
- This representation depends on basis choice.
- This lecture: matrix units $|i\rangle\langle j|$ for some ONB $\{|i\rangle\}$ of $\mathcal{H}$.
- vectorization mapping: $\quad \mathrm{vec}: \mathcal{B}(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}$
$$|i\rangle\langle j| \mapsto |i\rangle \otimes |j\rangle$$
+ linear extension
- Properties of vec:
  - isometry: $\langle X, Y\rangle_{\mathcal{B}(\mathcal{H})} = \langle \mathrm{vec}\, X, \mathrm{vec}\, Y\rangle_{\mathcal{H}\otimes\mathcal{H}}$
  - $\mathrm{vec}(|\psi\rangle\langle\varphi|) = |\psi\rangle \otimes |\bar{\varphi}\rangle$
  - $\mathrm{vec}(A X B^T) = (A \otimes B)\, \mathrm{vec}\, X$
- Linear map $\mathcal{N}(X) = \sum_i A_i X B_i^T \xmapsto{\mathrm{vec}} N = \sum_i A_i \otimes B_i$ transfer matrix
- $\mathcal{N}_3 = \mathcal{N}_2 \circ \mathcal{N}_1 \longrightarrow N_3 = N_2 N_1$ (channel composition $\to$ matrix mult.)
- quantum channels $\mathcal{N} = \sum_i K_i \cdot K_i^\dagger \xmapsto{\mathrm{vec}} N = \sum_i K_i \otimes \bar{K}_i$

§ 1.7 Complementary channels

$T: \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ quantum channel $(T: A \to B)$

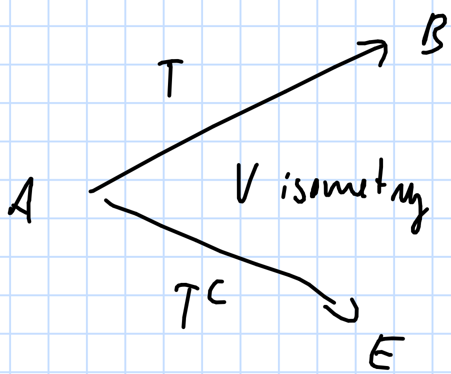
Prop 6: $\exists$ Stinespring isometry $V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ s.t.

$$T(X_A) = \mathrm{tr}_E\, V X_A V^\dagger$$



Noisy channel: environment E learns something about input X.

Complementary channel $\quad T^c(X_A) = \mathrm{tr}_B\, V X_A V^\dagger$



For given channel T with isometry V, the isometry $\underline{\tilde{V} = (\mathbb{1} \otimes U) V}$ with U unitary leads to the same channel T.

$$\left[\ \mathrm{tr}_E\left[(\mathbb{1}_B \otimes Y_E)\, X_{BE}\right] = \mathrm{tr}_E\left[X_{BE}\,(\mathbb{1}_B \otimes Y_E)\right]\ \right]$$

[cyclicity property of partial trace]

$$\Rightarrow \mathrm{tr}_E\, \tilde{V} X_A \tilde{V}^\dagger = \mathrm{tr}_E\left[(\mathbb{1}\otimes U)\, V X_A V^\dagger (\mathbb{1} \otimes U^\dagger)\right] = \mathrm{tr}_E\, V X_A V^\dagger = T(X_A)$$

$$\tilde V = (\mathbb{1}\otimes U)V: \quad \mathrm{tr}_B\, \tilde V X_A \tilde V^\dagger = \mathrm{tr}_B\left[(\mathbb{1}\otimes U) V X_A V^\dagger (\mathbb{1}\otimes U^\dagger)\right]$$

$$= U\, \mathrm{tr}_B(V X_A V^\dagger)\, U^\dagger$$

$$= U\, T^c(X_A)\, U^\dagger$$

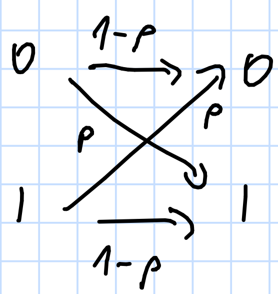
Ex.: related to $V = \sum_i U_i \otimes |i\rangle \leftarrow \{|i\rangle\}$ is <u>some</u> ONB

2. Classes of quantum channels

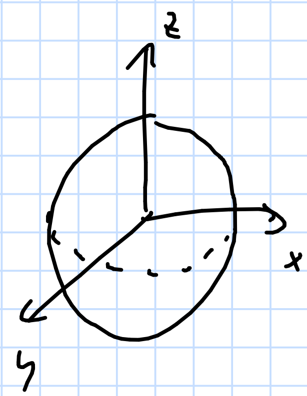
§2.1 Qubit channels

a) Bit-flip channel or X-dephasing channel

Classical information theory



Bloch sphere



Quantum version: Pauli-X operator $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

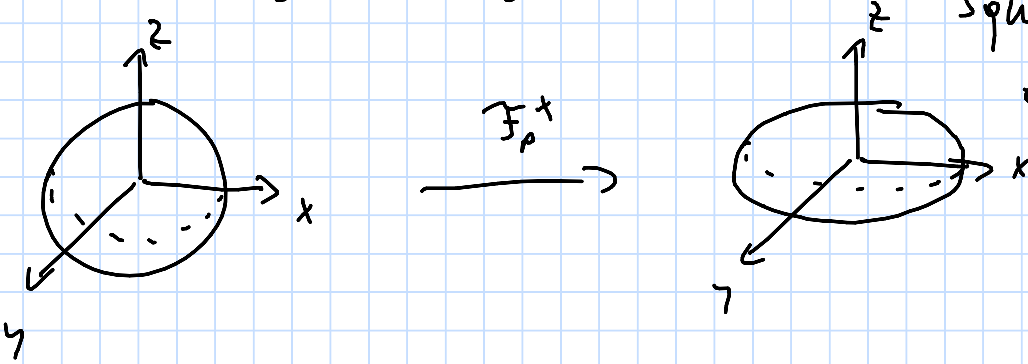
$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle$

eigenbasis: $|\pm\rangle := \dfrac{|0\rangle \pm |1\rangle}{\sqrt 2}$

$X|+\rangle = |+\rangle, \quad X|-\rangle = -|-\rangle$

$\mathcal{F}_p^X: \rho \mapsto (1-p)\rho + p X\rho X \qquad$ (Kraus op's: $\sqrt{1-p}\,\mathbb{1}, \sqrt{p}\, X$)

bit-flip channel $\mathcal{F}_p^x : \rho \mapsto (1-p)\rho + p X \rho X$



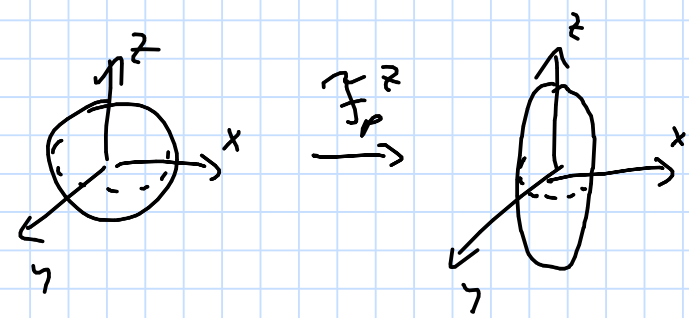
squash x and z-direction in Bloch sphere

b) Phase-flip channel or Z-dephasing channel

Pauli $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$

$Z|+\rangle = |-\rangle$, $Z|-\rangle = |+\rangle$

$\mathcal{F}_p^z : \rho \mapsto (1-p)\rho + p Z \rho Z$



c) Bit-phase flip channel or Y-dephasing channel

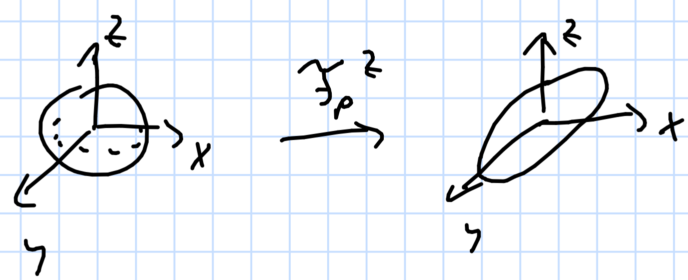
Pauli-$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ eigenbasis: $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$

$Y|+i\rangle = |+i\rangle$, $Y|-i\rangle = -|-i\rangle$

$XZ = -iY$

$Y|0\rangle = i|1\rangle$, $Y i|1\rangle = |0\rangle$

$\mathcal{F}_p^y : \rho \mapsto (1-p)\rho + p \underbrace{Y\rho Y}_{} = XZ\rho ZX$



$\mathcal{F}_p^*$ is unital for $* = X, Y, Z$: $X^2 = Y^2 = Z^2 = \mathbb{1}$

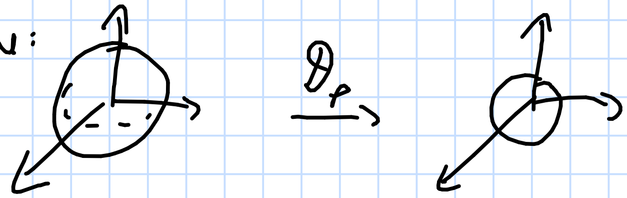
e.g. $\mathcal{F}_p^y(\mathbb{1}) = (1-p)\mathbb{1} + p \overbrace{Y\mathbb{1}Y}^{Y^2 = \mathbb{1}} = \mathbb{1}.$

d) Depolarizing channel

error model: each Pauli error (X, Y, Z) happens with equal probability $p/3$

$$\mathcal{D}_p : \rho \mapsto (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

Bloch sphere:



shrinks sphere <u>uniformly</u>

Kraus op's: $\sqrt{1-p}\,\mathbb{1}$, $\sqrt{\frac{p}{3}}\,X$, $\sqrt{\frac{p}{3}}\,Y$, $\sqrt{\frac{p}{3}}\,Z$: Kraus rank $r(\mathcal{D}_p) = 4$ for $p \in (0,1)$

Alternative representation: $\underline{\rho \mapsto (1-q)\rho + q\,\mathrm{tr}(\rho)\,\frac{1}{2}\mathbb{1}_2}$

replace input state ρ by completely mixed state $\frac{1}{2}\mathbb{1}_2 = \pi_2$ with "probability" q (q can be >1)

Relation $p \leftrightarrow q$?

$$\frac{1}{2}\mathbb{1}_2 = \frac{1}{4}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right) \quad \forall \rho \in B(\mathbb{C}^2)$$

$$(1-q)\rho + q\frac{\mathbb{1}}{2} = (1-q)\rho + \frac{q}{4}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right)$$

$$\Rightarrow q = \frac{4}{3}p \in \left[0, 4/3\right]$$

e) (generalized) Pauli channels

Let $\underline{p} = (p_0, p_1, p_2, p_3)$ be a probability distribution

$$\mathcal{N}_{\underline{p}}(\rho) = p_0 \rho + p_1 X \rho X + p_2 Y \rho Y + p_3 Z \rho Z \qquad \left(\text{depolarizing: } p_0 = 1-p, \; p_i = p/3\right)$$

$\forall \underline{p}$, the channel $\mathcal{N}_{\underline{p}}$ is <u>unital</u>.

Pauli channels are interesting (from an IT point of view):

a) classical information transmission (almost) <u>completely</u> understood!

b) quantum information transmission <u>not at all</u> understood!

(except for some special cases like flip/dephasing channels)