Recap

·) Quantum systems are modeled by Hilbert spaces.

·) Quantum state: $\rho \in B(\mathcal{H})$, $\rho \geq 0$, $\operatorname{tr}\rho = 1$.

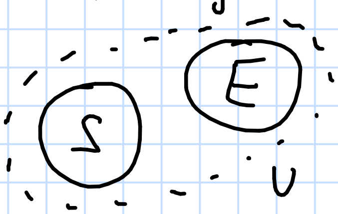
$\langle \psi_i | \psi_j \rangle = \delta_{ij}$

Eigendecomposition: $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, $\rho|\psi_i\rangle = \lambda_i |\psi_i\rangle$, $\lambda_i \geq 0$

·) Schrödinger picture: evolution of quantum states.

·) Closed systems: evolution given by unitary maps (Wigner's thm)

·) Open systems: unitary evolution on system + environment



induces noisy evolution on system.

·) Quantum channel: $T: B(\mathcal{H}_1) \to B(\mathcal{H}_2)$

$T \geq 0 \iff (X \geq 0 \Rightarrow T(X) \geq 0)$

-) linear

-) trace-preserving: $\operatorname{tr}(T(X)) = \operatorname{tr} X$ $\forall X \in B(\mathcal{H}_1)$

-) completely positive: $T \otimes \mathrm{id}_n \geq 0$ $\forall n \in \mathbb{N}$

·) Choi operator $B(\mathcal{H}_1 \otimes \mathcal{H}_2) \ni \tau := (\mathrm{id} \otimes T)(\gamma)$

where $|\gamma\rangle = \sum_i |i\rangle \otimes |i\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_1$

·) Choi-Jamiołkowski isomorphism: $T \mapsto \tau = (\mathrm{id} \otimes T)(\gamma)$

is a bijection $\{T: B(\mathcal{H}_1) \to B(\mathcal{H}_2) \text{ linear}\} \leftrightarrow B(\mathcal{H}_1 \otimes \mathcal{H}_2)$

with inverse $\tau \mapsto \left[T: X \mapsto \operatorname{tr}_1\left(\tau (X^T \otimes \mathbb{1})\right)\right]$.

·) Steering equality: $\forall |\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ $\exists K \in B(\mathcal{H}_1, \mathcal{H}_2)$: $|\psi\rangle = (\mathbb{1} \otimes K)|\gamma\rangle$

$T : B(\mathcal{H}_1) \to B(\mathcal{H}_2) \longrightarrow$ Choi operator $\tau = (id \otimes T)(\gamma)$

$|\gamma\rangle = \sum_i |i\rangle \otimes |i\rangle$

$\boxed{\text{Prop 3}}$ T, τ as above.

i) $T(X)^\dagger = T(X^\dagger)$ iff $\tau = \tau^\dagger$

ii) T is CP iff $\tau \geq 0$.

iii) T is TP iff $tr_2 \tau = \mathbb{1}_1$

iv) T is unital iff $tr_1 \tau = \mathbb{1}_2$.

Reminder: S is unital if $S(\mathbb{1}_1) = \mathbb{1}_2$.

Proof: i) ($\Rightarrow$) $\tau^\dagger = \tau$: $\tau^\dagger = \left(\sum_{i,j} |i\rangle\langle j|_1 \otimes T(|i\rangle\langle j|)\right)^\dagger$

$$= \sum_{i,j} |j\rangle\langle i| \otimes \underbrace{T(|i\rangle\langle j|)^\dagger}_{} = \tau$$

$\tau = \sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|)$ $\qquad T(|i\rangle\langle j|^\dagger) = T(|j\rangle\langle i|)$ ✓

($\Leftarrow$) $\tau^\dagger = \tau$: $T(|i\rangle\langle j|)^\dagger = T(|j\rangle\langle i|) \; \forall i,j$

$$T(X)^\dagger = T(X^\dagger) \;\; \forall X \in B(\mathcal{H}_1) \quad ✓$$

ii) ($\Rightarrow$) $\tau = (id \otimes T)(\gamma)$, $\gamma \geq 0 \Rightarrow \tau \geq 0$ by def.

($\Leftarrow$) to show: $(id_n \otimes T)(\rho) \geq 0 \quad \forall \rho \in B(\mathbb{C}^n \otimes \mathcal{H}_1), \rho \geq 0$

$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, $\lambda_i \geq 0$

Claim follows from $(id_n \otimes T)(\psi_i) \geq 0 \;\; \forall i$

Steering equality: $|\psi_i\rangle = (K_i \otimes \mathbb{1})|\gamma\rangle$, $K_i \in B(\mathcal{H}_1, \mathbb{C}^n)$

$\exists K_i : \quad |\psi_i\rangle = (K_i \otimes \mathbb{1}) |\gamma\rangle$

$$(id_n \otimes T)(\psi_i) = (id_n \otimes T)\left((K_i \otimes \mathbb{1}) |\gamma\rangle\langle\gamma| (K_i \otimes \mathbb{1})^\dagger \right)$$

$$= (K_i \otimes \mathbb{1}) \underbrace{(id_n \otimes T)(|\gamma\rangle\langle\gamma|)}_{= \tau \geq 0} (K_i \otimes \mathbb{1})^\dagger \geq 0$$

$\forall \psi_i \Rightarrow \forall \rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \Rightarrow CP.$

$\boxed{X \geq 0 \Rightarrow C X C^\dagger \geq 0 \quad \forall C}$

iii) T is TP $\Leftrightarrow tr_2 \tau = \mathbb{1}_1$

$\Rightarrow$ $tr_2 \tau = tr_2 \left(\sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|) \right) = \sum_{i,j} |i\rangle\langle j| \otimes \underbrace{tr(T(|i\rangle\langle j|))}_{tr |i\rangle\langle j| = \delta_{ij}}$

$$= \sum_i |i\rangle\langle i|_1 = \mathbb{1}_1 .$$

$\Leftarrow$ $\underset{\text{Prop 3}}{tr\, T(X)} = tr\left[tr_1(\tau (X^T \otimes \mathbb{1})) \right] = tr(\tau (X^T \otimes \mathbb{1}))$

$= tr(\underline{tr_2 \tau} \cdot X^T)$ Def of partial trace

$= tr X^T = \underline{tr X}$

iv) T is unital iff $tr_1 \tau = \mathbb{1}_2$

$\Rightarrow$ $\underline{tr_1 \tau} = \sum_{i,j} \underbrace{tr(|i\rangle\langle j|_1)}_{= \delta_{ij}} T(|i\rangle\langle j|_1) = \sum_i T(|i\rangle\langle i|_1)$

$$= T(\mathbb{1}_1) = \underline{\mathbb{1}_2}$$

$\Leftarrow$ $T(\mathbb{1}_1) = tr_1(\tau (\mathbb{1}_1^T \otimes \mathbb{1}_2)) = tr_1 \tau = \mathbb{1}_2$ $\square$

Examples of CP maps

a) unitary maps are CP : $(\mathbb{1}\otimes U)\,|\gamma\rangle\langle\gamma|\,(\mathbb{1}\otimes U)^\dagger \geq 0$

also TP!

a') isometries $V: \mathcal{H}_1 \to \mathcal{H}_2$: $V^\dagger V = \mathbb{1}_1$

$\Leftrightarrow \langle\varphi|V^\dagger V|\psi\rangle = \langle\varphi|\psi\rangle \;\forall \varphi,\psi$

$\dim \mathcal{H}_1 \leq \dim \mathcal{H}_2$

b) trace : $(\mathrm{id}_1 \otimes \mathrm{tr})(\gamma) = \sum_{i,j} |i\rangle\langle j|\, \underbrace{\mathrm{tr}(|i\rangle\langle j|)}_{=\delta_{ij}} = \mathbb{1}_1 \geq 0$

$\Rightarrow$ partial trace $\mathrm{tr}_2 : \mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$ is CPTP

$\Rightarrow \mathrm{tr}_E(V\rho V^\dagger)$ is CPTP (in fact, all CPTP maps have this form!)

c) $X \mapsto \sum_i K_i X K_i^\dagger$ CP

Example of a map that is positive, but not CP:

transposition $\vartheta : X \mapsto X^T$ (w.r.t. a fixed basis)

$\mathbb{F} = (\mathbb{1}\otimes\vartheta)(\gamma) = \sum_{i,j} |i\rangle\langle j| \otimes \vartheta(|i\rangle\langle j|) = \sum_{i,j} |i\rangle\langle j| \otimes |j\rangle\langle i|$

$|\varphi_1\rangle, |\varphi_2\rangle \in \mathcal{H}_1$: $\mathbb{F}(|\varphi_1\rangle\otimes|\varphi_2\rangle) = |\varphi_2\rangle\otimes|\varphi_1\rangle$ <u>swap operator</u>

$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$: $\mathbb{F}|\psi_-\rangle = -|\psi_-\rangle \Rightarrow \mathbb{F} \not\geq 0$

$\Rightarrow$ <u>ϑ is not CP</u>

§1.4 Kraus representation and isometric picture

We saw before: $X \mapsto \sum_i K_i X K_i^\dagger$ is CP

converse is also true:

$\boxed{\text{Prop 5}}$ a) A map $T: B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is CP iff $\exists \{K_i\}_i$ with $K_i \in B(\mathcal{H}_1, \mathcal{H}_2)$ s.t. $T(X) = \sum_i K_i X K_i^\dagger$.

K_i ... Kraus operators of T.

b) The Kraus rank $r(T)$, the minimal number of Kraus op's, is equal to the rank of the Choi op. $\tau = (id \otimes T)(\gamma)$.

$r(T) \leq \dim \mathcal{H}_1 \dim \mathcal{H}_2$

c) There exists a Kraus representation with $r = \operatorname{rank}(\tau)$ op's s.t. $\langle K_i, K_j \rangle = \operatorname{tr}(K_i^\dagger K_j) = \delta_{ij} c_i$.

d) T is TP iff $\sum_i K_i^\dagger K_i = \mathbb{1}_1$, T is unital iff $\sum_i K_i K_i^\dagger = \mathbb{1}_2$.

e) Any two Kraus rep's $\{K_i\}_i$, $\{L_i\}_i$ of a channel T are related by a unitary U: $K_i = \sum_{i,j} U_{ij} L_j$

$$\left(T(X) = \sum_i K_i X K_i^\dagger = \sum_j L_j X L_j^\dagger \right)$$

Proof: a) $\Leftarrow$ ✓

$\Rightarrow$ T is CP $\overset{\text{Prop 4}}{\Longleftrightarrow}$ $\tau = (id \otimes T)(\gamma) \geq 0$

$\Rightarrow \exists$ vectors $\{|\varphi_i\rangle\}_i$ s.t. $\tau = \sum_i |\varphi_i\rangle\langle\varphi_i| = \sum_i (\mathbb{1} \otimes K_i)|\gamma\rangle\langle\gamma|(\mathbb{1} \otimes K_i^\dagger)$

$\forall i: |\varphi_i\rangle = (\mathbb{1} \otimes K_i)|\gamma\rangle$

$= (id \otimes T)(\gamma)$ with $\underline{T = \sum_i K_i \cdot K_i^\dagger}$. Prop 3 $\Rightarrow$ ✓

b) clear from a): $r = \mathrm{rank}(\tau) \Rightarrow$ at least r pure states in a decomposition of τ.

$\Rightarrow$ at least r Kraus op's.