Quantum channels I: Representations & properties

Section 1: Representations

§1.1 Unitary evolution of (closed) quantum systems

Quantum mechanics

- quantum system: Hilbert space $\mathcal{H}$ $\quad(\dim \mathcal{H} < \infty)$
- Observables: measurable quantities

  Hermitian op's $A \in B(\mathcal{H})$, $A^\dagger = A$, $A = \sum_i \overset{\mathbb{R}}{\Downarrow}{a_i} \, |a_i\rangle\langle a_i|$ $\quad \langle a_i | a_j \rangle = \delta_{ij}$
- eigenvalues $a_i \in \mathbb{R}$ are the possible measurement outcomes.
- quantum states on $\mathcal{H}$: $\rho \in B(\mathcal{H})$, $\rho \geq 0$, $\operatorname{tr} \rho = 1 \iff \sum \lambda_i = 1$

  $\rho = \sum \lambda_i |\psi_i\rangle\langle\psi_i|$, $\rho|\psi_i\rangle = \lambda_i |\psi_i\rangle$
- pure quantum state has rank 1: $\exists\, |\psi\rangle \in \mathcal{H}$ s.t. $\psi = |\psi\rangle\langle\psi|$
- $\mathcal{H}$, state ρ on $\mathcal{H}$, observable A: $p_i = \operatorname{tr}(\rho\, |a_i\rangle\langle a_i|) = \langle a_i | \rho | a_i \rangle$
- Expected outcome of meas't: $\langle A \rangle_\rho = \sum_i p_i a_i = \operatorname{tr}(\rho A)$

Evolution of quantum systems

a) linear operations evolving states (Schrödinger pic), observables (Heisenberg pic), or both (interaction pic)

b) transformation $T: B(\mathcal{H}) \to B(\mathcal{H})$ should preserve $\mathcal{H}$-structure:

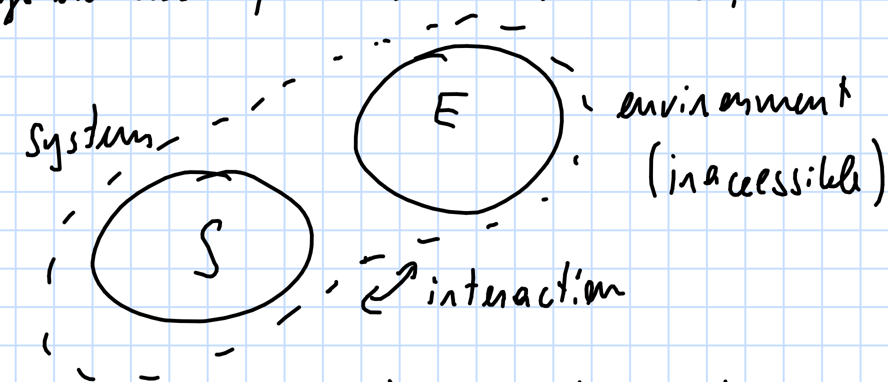
$$\operatorname{tr}(\psi \phi) = \operatorname{tr}(T(\psi) T(\psi)) \quad \forall\, \psi, \phi \in B(\mathcal{H})$$

Wigner's Thm: a + b $\Rightarrow$ $T(X) = U X U^\dagger$ or $T(X) = U X^T U^\dagger$ for unitary U

<u>unitary</u> anti-unitary

§ 1.2 Open systems and noisy evolution

Closed system assumption is NOT realistic!



SE = system + environment

unitary group

$$X_{SE} \mapsto U X_{SE} U^\dagger, \quad U \in \mathcal{U}(\underline{\mathcal{H}_S \otimes \mathcal{H}_E})$$

partial trace tr_E $\downarrow$ $\qquad\downarrow$

$$X_S = \mathrm{tr}_E(U X_{SE} U^\dagger)$$

Tracing out E induces a <u>noisy / irreversible</u> evolution of S.

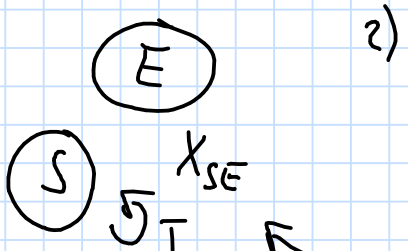
Schrödinger pic: evolve quantum states with map $T: B(\mathcal{H}_1) \to B(\mathcal{H}_2)$

<u>Requirements:</u> 1) Linearity

2) Map states to states

a) T should preserve trace: $\mathrm{tr}(T(X)) = \mathrm{tr}\, X$

~~b)~~ Positivity: $(X \geq 0 \Rightarrow T(X) \geq 0) \Leftrightarrow: T \geq 0$



b') Complete positivity: $\underline{T} \otimes \mathrm{id}_n \geq 0 \quad \forall n \in \mathbb{N}$

$\boxed{\text{Def 1}}$ A <u>quantum channel</u> is a linear, completely positive (CP), trace-preserving (TP) map $T: B(\mathcal{H}_1) \to B(\mathcal{H}_2)$.

$T: \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$: adjoint map $T^\dagger : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$

defined via $\quad \langle T^\dagger(X), Y \rangle = \langle X, T(Y) \rangle \quad \forall X \in \mathcal{B}(\mathcal{H}_2)$
$Y \in \mathcal{B}(\mathcal{H}_1)$

A linear map $T: \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ is:

·) CP iff $T^\dagger$ is CP.
·) TP iff $T^\dagger$ is unital: $T^\dagger(\mathbb{1}_2) = \mathbb{1}_1$.

$$\langle T^\dagger(\mathbb{1}_2), Y \rangle = \langle \mathbb{1}_2, T(Y) \rangle = \operatorname{tr}(T(Y)) = \operatorname{tr} Y = \langle \mathbb{1}_1, Y \rangle \quad \forall Y \in \mathcal{B}(\mathcal{H}_1)$$

$$\Rightarrow T^\dagger(\mathbb{1}_2) = \mathbb{1}_1$$

Unital quantum channels are both TP and unital.

§ 1.3 Choi - Jamiołkowski isomorphism

Very useful tool for studying quantum channels.

Def 2 Let $T: \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ be a linear map.

The Choi operator $\tau \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is defined as

$$\tau := (\mathrm{id}_1 \otimes T)(\gamma) \quad \text{where} \quad |\gamma\rangle = \sum_{i=1}^{\dim \mathcal{H}_1} |i\rangle \otimes |i\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_1$$

and $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_1}$ is an orthonormal basis for $\mathcal{H}_1$.

$$\tau = \sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|)$$

Ex.: $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$: $\quad \tau = \begin{pmatrix} T(|0\rangle\langle 0|) & T(|1\rangle\langle 0|) \\ T(|0\rangle\langle 1|) & T(|1\rangle\langle 1|) \end{pmatrix}$ block matrix

($T(|i\rangle\langle j|)$ are operators)

Prop 3 Let $T: B(\mathcal{H}_1) \to B(\mathcal{H}_2)$. The map $T \mapsto \tau = (id \otimes T)(\gamma)$ is a bijection between $\{T: B(\mathcal{H}_1) \to B(\mathcal{H}_2)\}$ and $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, with inverse mapping $\tau \mapsto T(X) := tr_1(\tau(X^T \otimes \mathbb{1}))$

<u>Proof:</u> Let $\tau = (id \otimes T)(\gamma)$

$$tr_1(\tau(X^T \otimes \mathbb{1})) = tr_1\left(\left(\sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|)\right)(X^T \otimes \mathbb{1})\right)$$

$$= \sum_{i,j} \underbrace{tr(|i\rangle\langle j| X^T)}_{\langle j|X^T|i\rangle} T(|i\rangle\langle j|)$$

$$= \langle i|X|j\rangle = x_{ij}$$

$$= \sum_{i,j} x_{ij}\, T(|i\rangle\langle j|) = T\Big(\overbrace{\sum_{i,j} x_{ij}\, |i\rangle\langle j|}^{X}\Big) = T(X) \checkmark$$

remains to be shown: $T \mapsto \tau = (id \otimes T)(\gamma)$ is surjective.

$$\exists\, |\psi_i\rangle, |\varphi_i\rangle \text{ s.t. } B(\mathcal{H}_1 \otimes \mathcal{H}_2) \ni \tau = \sum_i |\psi_i\rangle\langle\varphi_i| \qquad |\psi_i\rangle \neq |\varphi_i\rangle$$

<u>Claim:</u> For every vector $|\psi\rangle \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ $\exists\, V \in B(\mathcal{H}_1, \mathcal{H}_2)$ s.t. $\underline{|\psi\rangle} = (\mathbb{1}_1 \otimes V)\underline{|\gamma\rangle}$ ✓

<u>Proof:</u> $|\psi\rangle = \sum_{i,j} p_{ij}\, |i\rangle \otimes |e_j\rangle \longrightarrow V = \sum_{i,j} p_{ij}\, |e_j\rangle\langle i|$

γ-basis   arbitrary basis

$$\tau = \sum_i |\psi_i\rangle\langle\psi_i| \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

Claim $\Rightarrow \exists\, L_i, K_i$ s.t. $\quad |\psi_i\rangle = (\mathbb{1} \otimes K_i)|\gamma\rangle$

$$|\varphi_i\rangle = (\mathbb{1} \otimes L_i)|\gamma\rangle$$

$$\Rightarrow \quad \tau = \sum_i |\psi_i\rangle\langle\varphi_i| = \sum_i (\mathbb{1} \otimes K_i)|\gamma\rangle\langle\gamma|(\mathbb{1} \otimes L_i)^\dagger$$

$$\Rightarrow \quad \tau = (\mathrm{id} \otimes T)(\gamma) \quad \text{where } T(X) = \sum_i K_i X L_i^\dagger,$$

linear! $\square$