Math 595: Representation-theoretic methods in QIT (2022 fall term)

Exercise sheet 6

Last update: October 24, 2022

1. Let $\mathcal{H} \cong \mathbb{C}^d$ and denote by $\mathbb{F}$ the swap operator acting on $\mathcal{H} \otimes \mathcal{H}$.

   (a) Show that $\mathbb{F}$ can be written as
   $$\mathbb{F} = \sum_{i,j=1}^{d} |i\rangle\langle j| \otimes |j\rangle\langle i| \tag{1}$$
   for *any* ONB $\{|i\rangle\}_{i=1}^d$ of $\mathcal{H}$. Use this result to show that $\vartheta_2(\mathbb{F}) = d\Phi^+$, where
   $$|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle. \tag{2}$$

   (b) Prove the "transpose trick": $(X \otimes \mathbb{1})|\Phi^+\rangle = (\mathbb{1} \otimes X^T)|\Phi^+\rangle$ for any $X \in \mathcal{L}(\mathcal{H})$.

   (c) Prove the "swap trick": $\operatorname{tr}[\mathbb{F}(X \otimes Y)] = \operatorname{tr}(XY)$ for any $X, Y \in \mathcal{L}(\mathcal{H})$.

2. For Hermitian operators X, Y we write $X \geq Y :\Leftrightarrow X - Y \geq 0$.

   (a) Show that every mixed state ρ_A satisfies $\rho_A \leq \mathbb{1}_A$.

   (b) Use the above to show the so-called *reduction criterion:* every separable state ρ_{AB} satisfies $\rho_{AB} \leq \rho_A \otimes \mathbb{1}_B$ and $\rho_{AB} \leq \mathbb{1}_A \otimes \rho_B$.

   (c) Give an example of an entangled state violating the reduction criterion.

3. Let $\mathcal{H}_A, \mathcal{H}_B \cong \mathbb{C}^d$ and $\rho_{AB}(x) = (1-x)\Phi^+_{AB} + \frac{x}{d^2}\mathbb{1}_{AB}$ for $x \in [0, d^2/(d^2-1)]$ be an isotropic state.

   (a) Let σ_{AB} be an arbitrary state, and set $f = \operatorname{tr}(\sigma_{AB}\Phi^+_{AB})$. Show that
   $$\overline{\sigma}_{AB} = \int_{\mathcal{U}_d} dU\, (U \otimes \bar{U})\sigma_{AB}(U \otimes \bar{U})^\dagger \tag{3}$$
   is an isotropic state $\rho_{AB}(x)$ with $x = \frac{d^2}{d^2-1}(1-f)$.

   (b) Show that $\rho_{AB}(x)$ is entangled if $x < d/(d+1)$.
   Hint: Use the PPT-criterion, i.e., $\rho_{AB}(x)$ is entangled if $\vartheta_B(\rho_{AB}(x))$ has a negative eigenvalue.

   (c) Show that for every $x \in [d/(d+1), d^2/(d^2-1)]$ the isotropic state $\rho_{AB}(x)$ is separable.
   Hint: Show this by constructing for every $x \in [d/(d+1), d^2/(d^2-1)]$ a product state $\sigma_{AB} = \chi_A \otimes \omega_B$ such that $\overline{\sigma}_{AB}$ as defined in (3) is a separable isotropic state with parameter x.

   (d) Conclude that $\rho_{AB}(x)$ is separable if and only if $x \in [d/(d+1), d^2/(d^2-1)]$.