Math 595: Representation-theoretic methods in QIT (2022 fall term)

Exercise sheet 5

Last update: November 1, 2022

1. Let ρ be a mixed state of rank r, and consider a spectral decomposition $\rho = \sum_{i=1}^{r} \lambda_i |v_i\rangle\langle v_i|$, where $\lambda_i > 0$ are the non-zero eigenvalues of ρ with corresponding eigenvector $|v_i\rangle$. Further, for some $k \geq r$ let $\mathcal{V}\colon \mathbb{C}^r \to \mathbb{C}^k$ be an isometry, i.e., $\mathcal{V}^\dagger \mathcal{V} = \mathbb{1}_{\mathbb{C}^k}$, and denote by $V \in M_{r,k}(\mathbb{C})$ its matrix representation with respect to fixed ONBs on $\mathbb{C}^r$ and $\mathbb{C}^k$. Define the *unnormalized states*

$$|\tilde{\psi}_i\rangle = \sum_{j=1}^{k} V_{ij} \sqrt{\lambda_j} |v_j\rangle \quad \text{for } i = 1, \ldots, k. \tag{1}$$

Show that

$$\rho = \sum_{i=1}^{k} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_{i=1}^{k} p_j |\psi_i\rangle\langle\psi_i|, \tag{2}$$

where we defined $p_i = \langle\tilde{\psi}_i|\tilde{\psi}_i\rangle$ and $|\psi_i\rangle = p_i^{-1/2}|\tilde{\psi}_i\rangle$.

2. Show that the following operators form a POVM on $\mathbb{C}^2$ with ONB $\{|0\rangle, |1\rangle\}$:

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1| \qquad E_2 = \frac{\sqrt{2}}{2+2\sqrt{2}}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|) \qquad E_3 = \mathbb{1} - E_1 - E_2. \tag{3}$$

Suppose that Alice gives Bob a system prepared in either

$$|\psi_1\rangle = |0\rangle \qquad \text{or} \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle). \tag{4}$$

Discuss the possible measurement outcomes and conclude that Bob never makes the error of misidentifying the state when using the POVM $\{E_1, E_2, E_3\}$.

3. Let $\mathcal{H}$ be a Hilbert space of dimension $\dim \mathcal{H} = d$. Show that a collection of d vectors $|v_1\rangle, \ldots, |v_d\rangle$ is an orthonormal basis if and only if $\sum_{j=1}^{d} |v_j\rangle\langle v_j| = \mathbb{1}_{\mathcal{H}}$.

4. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces with orthonormal bases $\{|a_i\rangle_A\}_{i=1}^{|A|}$ and $\{|b_i\rangle_B\}_{i=1}^{|B|}$, respectively. Let $|\psi\rangle_{AB} = \sum_{i=1}^{|A|}\sum_{j=1}^{|B|} x_{ij}|a_i\rangle_A \otimes |b_j\rangle_B$ with $x_{ij} \in \mathbb{C}$ satisfying $\sum_{i,j}|x_{ij}|^2 = 1$ be an arbitrary pure bipartite state. Define the matrix $X \in M_n(\mathbb{C})$ ($n = |A||B|$) with components $(X)_{ij} = x_{ij}$. Use the singular value decomposition of X to prove the Schmidt decomposition theorem from the lecture.

5. Use the Schmidt decomposition theorem to show that any two purifications of a mixed state are related by an isometry acting on the purifying system.