Math 595: Representation-theoretic methods in QIT (2022 fall term)

Exercise sheet 4

Last update: November 1, 2022

1. Let V be a vector space and $\mathcal{A} \leq \mathrm{End}(V)$ be an algebra of operators. Show that a projection $p \in \mathcal{A}$ is minimal iff it has rank 1.

2. Let G be a finite group and $\mathcal{R}(G)$ its regular representation, which decomposes as

$$\mathcal{R}(G) \cong \bigoplus_{\alpha} V_\alpha \otimes \mathbb{C}^{d_\alpha},$$

where $\{V_\alpha\}_\alpha$ is a complete set of inequivalent irreducible representations of G, and $d_\alpha = \dim V_\alpha$. One can show (cf. Teleman's lecture notes, Thm. 9.9) that the group algebra $\mathbb{C}[G]$ has an analogous decomposition

$$\mathbb{C}[G] \cong \bigoplus_{\alpha} \mathrm{End}(\mathbb{C}^{d_\alpha}). \tag{1}$$

Use this together with Ex. 1 to show that minimal projections in $\mathbb{C}[G]$ are in $1:1$ correspondence with equivalence classes of irreducible representations of G.

Hint: The decomposition (1) shows that every $x \in \mathbb{C}[G]$ can be written as $x = \bigoplus_\alpha x_\alpha$ for some $x_\alpha \in \mathrm{End}(\mathbb{C}^{d_\alpha})$. Now apply this to projections and use Ex. 1 to argue about minimality.

3. Let $\mathcal{A}$ be an algebra with decomposition $\mathcal{A} \cong \bigoplus_i \mathcal{A}_i$. Show that $\mathcal{Z}(\mathcal{A}) \cong \bigoplus_i \mathcal{Z}(\mathcal{A}_i)$.

4. Let $(i_1, \ldots, i_k)$ with $k \leq n$ be a cycle in S_n, and let $\pi \in S_n$ be arbitrary. Show that

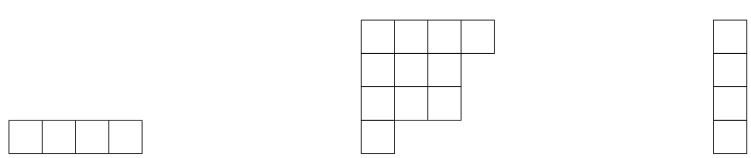
$$\pi(i_1, \ldots, i_k)\pi^{-1} = (\pi(i_1), \ldots, \pi(i_k)).$$

5. We denote by

$$P_a = \frac{1}{n!} \sum_{\pi \in S_n} \mathrm{sgn}(\pi)\pi \in \mathbb{C}[G]$$

the projector onto the antisymmetric subspace. Show that, if $d < n$, then $P_a(\mathbb{C}^d)^{\otimes n} = 0$.

6. Compute the dimensions of the irreducible representations V_λ and U_λ of S_n and $\mathcal{U}_d$ for the following Young diagrams $\lambda \vdash_d n$:



7. Compute the dimensions of the irreducible representation V_λ of S_n for $\lambda = (n, 0, \ldots, 0) \vdash_n n$ and $\lambda = (1, \ldots, 1) \vdash_n n$.