Math 595: Representation-theoretic methods in QIT (2022 fall term)

Exercise sheet 3

Last update: October 3, 2022

1. Let G and H be groups with irreducible representations (φ, V) and (ψ, W), respectively. Show that the external product representation $\varphi \hat{\otimes} \psi \colon (g, h) \longmapsto \varphi(g) \otimes \psi(h)$ on $V \otimes W$ is an irreducible representation of the direct product group $G \times H$.

2. Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$, and let $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} \colon ab = ba \text{ for all } b \in \mathcal{A}\}$ be its center. Show that $\mathcal{Z}(\mathcal{A})$ is an $\mathbb{F}$-subalgebra of $\mathcal{A}$.

3. Let $M_d(\mathbb{C})$ be the algebra of $(d \times d)$-matrices over $\mathbb{C}$. Prove that

$$\mathcal{Z}(M_d(\mathbb{C})) \cong \mathbb{C}.$$

   Hint: Either use Schur's Lemma, or prove this directly using elementary matrices E_{ij}.

4. Let V and W be vector spaces over the same field. Show that $\operatorname{End}(V \otimes W) \cong \operatorname{End}(V) \otimes \operatorname{End}(W)$.

5. Let (φ, V) and (ψ, W) be representations of a finite group G and let $f \in \hom(V, W)$ be an arbitrary linear map. Define the linear map

$$f_G = \frac{1}{|G|} \sum_{g \in G} \psi(g) \circ f \circ \varphi(g)^{-1} \in \hom(V, W). \tag{1}$$

   Show the following statements using Schur's Lemma:

   (a) f_G is an intertwiner, i.e., $f_G \circ \varphi(h)^{-1} = \psi(h) \circ f_G$ for all $h \in G$.

   (b) If φ and ψ are irreducible and inequivalent, then $f_G = 0$.

   (c) If $(\varphi, V) = (\psi, W)$ is irreducible, then $f_G = \frac{\operatorname{tr} f}{\dim V} \mathbb{1}_V$.

   (d) Fix bases $\{|e_i\rangle\}_{i=1}^{\dim V}$ for V and $\{|f_i\rangle\}_{i=1}^{\dim W}$ for W, and denote by $\varphi_{ij} \colon G \to \mathbb{C}$ and $\psi_{ij} \colon G \to \mathbb{C}$ the corresponding matrix coefficients. Use the previous results to show that, if φ and ψ are irreducible,

$$\frac{1}{|G|} \sum_{g \in G} \psi_{ij}(g) \varphi_{kl}(g^{-1}) = \begin{cases} 0 \text{ for all } i, j, k, l & \text{if } \varphi \ncong \psi; \\ \frac{1}{\dim V} \delta_{il} \delta_{jk} & \text{if } \varphi = \psi. \end{cases}$$

   Hint: Expand the maps f and f_G in (1) in terms of the bases $\{|e_i\rangle\}_{i=1}^{\dim V}$ and $\{|f_i\rangle\}_{i=1}^{\dim W}$.