Math 595: Representation-theoretic methods in QIT (2022 fall term)

Exercise sheet 1

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1. Let G be a group. Show that the neutral element $e \in G$ and the inverse of any element $g \in G$ are unique.

2. Let G, H be groups and $\phi\colon G \to H$ be a group homomorphism, i.e., a map satisfying $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ for all $g_1, g_2 \in G$. Denoting the neutral elements in G and H by e_G and e_H, respectively, show that $\phi(e_G) = e_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.

3. Show that a group G acts on itself by left multiplication. How can you turn right multiplication into a valid group action?

4. Let X be an arbitrary set and G be a group acting on X. Denote this action by $g \cdot x \in X$ for $g \in G$ and $x \in X$. Consider the set $\mathbb{C}^X = \{f\colon X \to \mathbb{C}\}$ of complex-valued functions on X. Show that $(g \cdot f)(x) := f(g^{-1} \cdot x)$ for $g \in G$, $f \in \mathbb{C}^X$ and $x \in X$ defines a G-action on $\mathbb{C}^X$.

5. Let G be a finite group with representation (φ, W). Assume that there exists a vector $|w\rangle \in W$ such that the set $\{\varphi(g)|w\rangle\}_{g \in G}$ is a basis for W. Show that (φ, W) is isomorphic to the regular representation of W.

6. Let (φ_1, V_1) and (φ_2, V_2) be representations of a group G, and let $f\colon V_1 \to V_2$ be a G-equivariant linear map. Show that $\ker f \leq V_1$ and $\operatorname{im} f \leq V_2$ are G-invariant subspaces.

7. Let $V = \mathbb{C}^3$ with the standard basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$, and consider the representation (φ, V) of the symmetric group S_3 on V defined by permuting coordinates: for $|x\rangle = x_1|e_1\rangle + x_2|e_2\rangle + x_3|e_3\rangle$ with $x_i \in \mathbb{C}$, the group element $\pi \in S_3$ acts on V via

$$\varphi(\pi)|x\rangle = x_{\pi^{-1}(1)}|e_1\rangle + x_{\pi^{-1}(2)}|e_2\rangle + x_{\pi^{-1}(3)}|e_3\rangle.$$

   Determine a decomposition of (φ, V) into irreducible representations.

   Hint: First, try to find a 1-dimensional subspace on which S_3 acts trivially. Then, take a closer look at the orthogonal complement (with respect to the standard inner product on V) of this subspace. Are there any more invariant subspaces?

8. Use Schur's Lemma to show that every complex irreducible representation of an Abelian group is 1-dimensional.