Chapter 6: Invariant States

§ 6.1 Werner states

Detecting entanglement in arbitrary bipartite quantum states is hard! (In fact, NP-hard)

This task becomes easier when we impose symmetries.

$\boxed{\text{Def}}$ (Werner states)

Let $\mathcal{H}_A = \mathcal{H}_B \cong \mathbb{C}^d$ be d-dimensional Hilbert spaces, $d \geq 2$.

A quantum state ρ_{AB} on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called a **Werner state** if

$$(U \otimes U)\, \rho_{AB}\, (U \otimes U)^\dagger = \rho_{AB} \quad \text{for all } U \in \mathcal{U}_d$$

Recall: Schur-Weyl duality $\quad (\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} V_\lambda \otimes W_\lambda$

where: V_λ is an irrep of S_n of dim. $d_\lambda = \dfrac{n!}{\prod_{(i,j)\in\lambda} h(i,j)}$

W_λ is an irrep of $\mathcal{U}_d$ of dim. $m_\lambda = \prod_{1 \leq i < j \leq d} \dfrac{\lambda_j - \lambda_i + j - i}{j - i}$

Here: $\quad d_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = d_{\square\square} = 1 \quad \Rightarrow (\mathbb{C}^d)^{\otimes 2} = W_{\square\square} \oplus W_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

Symmetric subspace:

swap operator

$$W_{\Box\Box} = \mathrm{Sym}^2(\mathbb{C}^d) = \{ |v\rangle \in (\mathbb{C}^d)^{\otimes 2} : \mathbb{F}|v\rangle = |v\rangle \}$$

$$m_{\Box\Box} = \dim \mathrm{Sym}^2(\mathbb{C}^d) = \frac{d(d+1)}{2}$$

Antisymmetric subspace:

$$W_{\substack{\Box\\\Box}} = \mathrm{Alt}^2(\mathbb{C}^d) = \{ |v\rangle \in (\mathbb{C}^d)^{\otimes 2} : \mathbb{F}|v\rangle = -|v\rangle \}$$

$$m_{\substack{\Box\\\Box}} = \dim \mathrm{Alt}^2(\mathbb{C}^d) = \frac{d(d-1)}{2}$$

Schur's lemma: $(U\otimes U)\rho_{AB}(U\otimes U)^\dagger = \rho_{AB}$ implies that

$$\rho_{AB} = c_{\Box\Box}\,\mathbb{1}_{W_{\Box\Box}} \oplus c_{\substack{\Box\\\Box}}\,\mathbb{1}_{W_{\substack{\Box\\\Box}}} \quad \text{for some } c_{\Box\Box}, c_{\substack{\Box\\\Box}} \geq 0$$

$$\text{with } 1 = c_{\Box\Box}\frac{d(d+1)}{2} + c_{\substack{\Box\\\Box}}\frac{d(d-1)}{2}$$

The Young symmetrizers for $\Box\Box$ and $\substack{\Box\\\Box}$ are given by

$$e_{\Box\Box} = \mathbb{1} + \mathbb{F}, \qquad e_{\substack{\Box\\\Box}} = \mathbb{1} - \mathbb{F},$$

and hence we have the projectors

$$P_{\Box\Box} = \frac{1}{2}(\mathbb{1} + \mathbb{F}) \quad \text{onto} \quad V_{\Box\Box} \otimes W_{\Box\Box} = W_{\Box\Box}$$

1-dim

$$P_{\substack{\Box\\\Box}} = \frac{1}{2}(\mathbb{1} - \mathbb{F}) \quad \text{onto} \quad V_{\substack{\Box\\\Box}} \otimes W_{\substack{\Box\\\Box}} = W_{\substack{\Box\\\Box}}$$

$$\mathrm{tr}\, P_{\Box\Box} = \frac{1}{2}d(d+1), \qquad \mathrm{tr}\, P_{\substack{\Box\\\Box}} = \frac{1}{2}d(d-1)$$

Prop A Werner state has the form

$$\rho_{AB} = x \frac{2}{d(d+1)} P_{\square\square} + (1-x) \frac{2}{d(d-1)} P_{\boxminus}$$

where $x \in [0,1]$ and $P_{\square\square} = \frac{1}{2}(\mathbb{1} + \mathbb{F})$, $P_{\boxminus} = \frac{1}{2}(\mathbb{1} - \mathbb{F})$.

Proof: We have $\operatorname{tr} P_{\square\square} = \dim W_{\square\square} = \frac{d(d+1)}{2}$ and

$$\operatorname{tr} P_{\boxminus} = \dim W_{\boxminus} = \frac{d(d-1)}{2} \Rightarrow \text{claim.} \quad \square$$

Alternative parametrization:

$$\rho_{AB} = \frac{1}{d(d^2-1)} \left[(d-\alpha)\, \mathbb{1} + (d\alpha - 1)\, \mathbb{F} \right]$$

with the visibility $\alpha = \operatorname{tr}(\rho_{AB} \mathbb{F})$.

Def (Twirling operation)

For $X \in \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d)$ we define the twirling operation

$$\mathcal{T}(X) = \int_{\mathcal{U}_d} dU \, (U \otimes U) X (U \otimes U)^\dagger,$$

where dU denotes the Haar measure on $\mathcal{U}_d$.

Prop i) Every Werner state is invariant under $\mathcal{T}$.

ii) Let ρ_{AB} be an arbitrary state. Then $\mathcal{T}(\rho_{AB})$ is a Werner state of visibility $\alpha = \operatorname{tr}(\mathbb{F} \rho_{AB})$.

<u>Proof</u>: i) If $(U\otimes U)\varrho_{AB}(U\otimes U)^\dagger = \varrho_{AB}$ for all $U\in\mathcal{U}_d$, then
$$\mathcal{T}(\varrho_{AB}) = \int_{\mathcal{U}_d} dU\,(U\otimes U)\varrho_{AB}(U\otimes U)^\dagger$$
$$= \int_{\mathcal{U}_d} dU\,\varrho_{AB} = \varrho_{AB}$$
by normalization of the Haar measure.

ii) We compute:
$$(U\otimes U)\,\mathcal{T}(\varrho_{AB})\,(U\otimes U)^\dagger$$
$$= (U\otimes U)\left[\int_{\mathcal{U}_d} dV\,(V\otimes V)\varrho_{AB}(V\otimes V)^\dagger\right](U\otimes U)^\dagger$$
$$= \int_{\mathcal{U}_d} dV\,(UV\otimes UV)\varrho_{AB}(UV\otimes UV)^\dagger = \mathcal{T}(\varrho_{AB})$$
by left invariance of the Haar measure. Hence, $\mathcal{T}(\varrho_{AB})$ is a Werner state of visibility
$$\alpha = \mathrm{tr}\left(\mathcal{T}(\varrho_{AB})\,\mathbb{F}\right)$$
$$= \int_{\mathcal{U}_d} dU\,\mathrm{tr}\left[(U\otimes U)\varrho_{AB}(U\otimes U)^\dagger\,\mathbb{F}\right]$$
$$= \int_{\mathcal{U}_d} dU\,\mathrm{tr}\Big[\varrho_{AB}\,\underbrace{(U\otimes U)^\dagger\,\mathbb{F}\,(U\otimes U)}_{=\mathbb{F}\ \forall U}\Big]$$
$$= \mathrm{tr}(\varrho_{AB}\,\mathbb{F}).$$
$\square$

$\boxed{\text{Lem}}$ Let σ_{AB} be a separable state. Then $\mathcal{T}(\sigma_{AB})$ is separable as well, and $\mathrm{tr}\left(\mathcal{T}(\sigma_{AB})\,\mathbb{F}\right) \geq 0$.

Proof: If σ_{AB} is separable, i.e., $\sigma_{AB} = \sum_i p_i\, \sigma_A^{(i)} \otimes \sigma_B^{(i)}$, then clearly $(U\otimes U)\,\sigma_{AB}\,(U\otimes U)^\dagger$ is separable for all $U \in \mathcal{U}_d$, and a suitable approximation of the Haar integral using Riemann sums shows that $\mathcal{T}(\sigma_{AB}) = \int_{\mathcal{U}_d} dU\,(U\otimes U)\,\sigma_{AB}\,(U\otimes U)^\dagger$ is a limit of a convex combination of separable states and hence separable itself.

Now, for any product state $\rho_A \otimes \gamma_B$, a simple calculation shows that $\mathrm{tr}\left((\rho_A\otimes\gamma_B)\,\mathbb{F}\right) = \mathrm{tr}\left(\rho_A\cdot\gamma_B\right) \geq 0$, since $\rho_A, \gamma_B \geq 0$. Hence, $\mathrm{tr}\left(\sigma_{AB}\,\mathbb{F}_{AB}\right) = \sum_i p_i\, \mathrm{tr}\left[(\sigma_A^{(i)} \otimes \sigma_B^{(i)})\,\mathbb{F}\right] \geq 0$. $\square$

Remark: i) The useful identity $\mathrm{tr}\left[(X_A \otimes Y_B)\,\mathbb{F}\right] = \mathrm{tr}(X_A Y_B)$ is often called "swap trick".

ii) Let $\alpha \in [0,1]$ be arbitrary, and set $|\varphi\rangle = \sqrt{\alpha}\,|0\rangle + \sqrt{1-\alpha}\,|1\rangle$ for some orthonormal $|0\rangle, |1\rangle \in \mathbb{C}^d$. Then

$$\mathrm{tr}\left[(\varphi_A \otimes |0\rangle\langle 0|_B)\,\mathbb{F}\right] = \mathrm{tr}\left(|\varphi\rangle\langle\varphi|_A\, |0\rangle\langle 0|_A\right) = |\langle\varphi|0\rangle|^2 = \alpha,$$

and hence $\mathcal{T}(\varphi_A \otimes |0\rangle\langle 0|_B)$ is a sep. Werner state of visibility α.

We show now that every Werner state ρ_{AB} with $\mathrm{tr}(\rho_{AB}\mathbb{F})<0$ is entangled. To this end, we will employ a useful criterion for entanglement based on the partial transpose $\vartheta_B := \mathrm{id}_A \otimes \vartheta$, where $\vartheta: X \mapsto X^T$ denotes the transpose.

On product operators, $\vartheta_B(X_A \otimes Y_B) = X_A \otimes Y_B^T$.

$\boxed{\text{Prop}}$ (Positive partial transpose (PPT) criterion)

$\vartheta_B(\sigma_{AB}) \geq 0$ for every separable state σ_{AB}. Hence, if $\vartheta_B(\rho_{AB})$ has a negative eigenvalue, then ρ_{AB} is entangled.

Proof: Let $\sigma_{AB} = \sum_i p_i \sigma_A^{(i)} \otimes \sigma_B^{(i)}$ be separable. Since ϑ_B is linear and $X \geq 0 \Leftrightarrow X^T \geq 0$, we have

$$\vartheta_B(\sigma_{AB}) = \sum_i p_i \sigma_A^{(i)} \otimes \left(\sigma_B^{(i)}\right)^T \geq 0$$

as a convex combination of positive semidefinite operators. $\square$

$\boxed{\text{Lem}}$ A Werner state ρ_{AB} is entangled if $\mathrm{tr}(\rho_{AB}\mathbb{F})<0$.

Proof: With $\alpha = \mathrm{tr}(\rho_{AB}\mathbb{F}) < 0$, we can write

$$\rho_{AB} = \frac{1}{d(d^2-1)}\left[(d-\alpha)\mathbb{1} + (d\alpha - 1)\mathbb{F}\right].$$

We have: $\cdot)$ $\vartheta_B(\mathbb{1}_{AB}) = \mathbb{1}_{AB}$

$\cdot)$ Fixing an ONB $\{|i\rangle\}_{i=1}^{d}$ of $\mathbb{C}^d$, we have

$$\mathbb{F}_{AB} = \sum_{i,j=1}^{d} |i\rangle\langle j|_A \otimes |j\rangle\langle i|_B$$

and hence $\vartheta_B(\mathbb{F}_{AB}) = \sum_{i,j=1}^{d} |i\rangle\langle j|_A \otimes (|i\rangle\langle j|_B)^T = \sum_{i,j=1}^{d} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B$

$$= d\, |\phi^+\rangle\langle\phi^+|_{AB}$$

with $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_A |i\rangle_B$ a maximally entangled state.

Then, $\vartheta_B(\rho_{AB}) \propto (d-\alpha)\mathbb{1} + d(d\alpha - 1)|\phi^+\rangle\langle\phi^+| =: X_{AB}$

Let P_{AB} be the projection onto $\operatorname{span}\{|\phi^+\rangle\}^\perp$, then

$$\mathbb{1}_{AB} = |\phi^+\rangle\langle\phi^+|_{AB} + P,$$

and hence X_{AB} has eigenvalues

$$\lambda_1 = d - \alpha + d^2\alpha - d = \alpha(d^2-1)$$

$$\lambda_2 = d - \alpha.$$

For $\alpha < 0$ we have $\lambda_1 = \alpha(d^2-1) < 0$ (recall $d \geq 2$). $\square$

In summary, we have proved:

$\boxed{\text{Prop}}$ A Werner state ρ_{AB} is entangled if and only if $\operatorname{tr}(\rho_{AB}\mathbb{F})<0$.

Caution: The PPT criterion is generally only a necessary criterion for separability. There are entangled states ρ_{AB} with $\vartheta_B(\rho_{AB})\geq 0$, the so-called "bound entangled" states.

However: ·) A Werner state is separable iff it is PPT.

·) Let $|A|, |B|$ be such that $|A|\cdot|B|\leq 6$. Then a state is separable iff it is PPT.

·) This result can be generalized to some low-rank states in higher dimensions.

We can generalize Werner states to the multipartite setting:

Let $\mathcal{H}_{A_i}=\mathbb{C}^d$, $i=1,\dots,n$. A state $\rho_{A_1\dots A_n}$ is called a multipartite Werner state if

$$U^{\otimes n}\rho_{A_1\dots A_n}(U^\dagger)^{\otimes n}=\rho_{A_1\dots A_n}\qquad\text{for all } U\in\mathcal{U}_d.$$

Schur-Weyl duality: let $\mathcal{A} = \mathrm{span}\{U^{\otimes n} : U \in \mathcal{U}_d\}$ and

$\mathcal{B} = \mathrm{span}\{Q_\pi : \pi \in S_n\}$ ($Q_\pi := \varphi(\pi)$ in previous notation)

Then $U^{\otimes n} \rho_{A_1 \dots A_n} (U^\dagger)^{\otimes n} = \rho_{A_1 \dots A_n}$ for all $U \in \mathcal{U}_d$

$\Rightarrow \rho_{A_1 \dots A_n} \in \mathcal{A}' = \mathcal{B}$

$\Rightarrow \rho_{A_1 \dots A_n} = \sum_{\pi \in S_n} c_\pi Q_\pi$ for some $c_\pi \in \mathbb{C}$.

Compare $n=2$ case: $\rho_{A_1 A_2} = \alpha \mathbb{1} + \beta \mathbb{F}$.

Not always useful because the Q_π are not PSD!

Alternative: $(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda \vdash_d n} V_\lambda \otimes W_\lambda$

Schur's lemma: $U^{\otimes n}$-invariance forces $\rho_{A_1 \dots A_n}$ to be $\propto \mathbb{1}_{W_\lambda}$ on W_λ.

$$\Rightarrow \quad \rho_{A_1 \dots A_n} = \bigoplus_{\lambda \vdash_d n} x_\lambda \, \rho_\lambda \otimes \frac{1}{m_\lambda} \mathbb{1}_{W_\lambda}$$

where $(x_\lambda)_{\lambda \vdash_d n}$ is a prob. dist., ρ_λ is a quantum state on V_λ for $\lambda \vdash_d n$, and $m_\lambda = \dim W_\lambda$.

If in addition $\rho_{A_1 \dots A_n}$ is symmetric, $Q_\pi \rho_{A_1 \dots A_n} Q_\pi^\dagger = \rho_{A_1 \dots A_n}$ for all $\pi \in S_n$, then

$$\rho_{A_1 \dots A_n} = \bigoplus_{\lambda \vdash_d n} x_\lambda \frac{1}{d_\lambda} \mathbb{1}_{V_\lambda} \otimes \frac{1}{m_\lambda} \mathbb{1}_{W_\lambda} = \sum_{\lambda \vdash_d n} x_\lambda \, \rho_\lambda$$

where $\rho_\lambda = \frac{1}{d_\lambda m_\lambda} P_\lambda$, and P_λ is the projector onto $V_\lambda \otimes W_\lambda$.

§6.2 Isotropic states

For an ONB $\{|i\rangle\}_{i=1}^{d}$ of $\mathbb{C}^d$, consider the maximally entangled state $|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle$.

Lem (Transpose trick)

For any $X \in \mathcal{L}(\mathbb{C}^d)$, $(X \otimes \mathbb{1})|\phi^+\rangle = (\mathbb{1} \otimes X^T)|\phi^+\rangle$

Proof: Exercise. $\square$

It follows that for any unitary $U \in \mathcal{U}_d$ we have

$$(U \otimes \overline{U})|\phi^+\rangle = |\phi^+\rangle,$$

since $U^T \overline{U} = (\overline{U}^T U)^T = (U^\dagger U)^T = \mathbb{1}^T = \mathbb{1}$.

Def (Isotropic states)

A state ρ_{AB} on systems AB with $\mathcal{H}_A \cong \mathcal{H}_B \cong \mathbb{C}^d$ is called isotropic if $(U \otimes \overline{U})\rho_{AB}(U \otimes \overline{U})^\dagger = \rho_{AB}$ for all $U \in \mathcal{U}_d$.

Clearly, $(U \otimes \overline{U})\mathbb{1}_{AB}(U \otimes \overline{U})^\dagger = \mathbb{1}_{AB}$ for all $U \in \mathcal{U}_d$ as well.

Using $\vartheta_B(\phi^+_{AB}) = \frac{1}{d} F_{AB}$, Schur-Weyl duality shows:

Prop An isotropic state ρ_{AB} can be written as

$$\rho_{AB} = (1-x)|\phi^+\rangle\langle\phi^+|_{AB} + x \frac{1}{d^2} \mathbb{1}_{AB} \quad \text{for } x \in \left[0, \frac{d^2}{d^2-1}\right]$$

Since $\mathrm{tr}_B \, \phi_{AB} = \frac{1}{d} \mathbb{1}_A$, we have

$$\rho_{AB} = (1-x)\, \phi^+_{AB} + x \frac{1}{d^2} \mathbb{1}_A \otimes \mathbb{1}_B$$

$$= (id_A \otimes \mathcal{D}_x)(\phi^+_{AB}),$$

where we defined the depolarizing channel

$$\mathcal{D}_x(\omega) := (1-x)\,\omega + x \cdot \mathrm{tr}(\omega) \frac{1}{d} \mathbb{1},$$

an important noise model satisfying $\mathcal{D}_x(U \omega U^\dagger) = U \mathcal{D}_x(\omega) U^\dagger$

for all $\omega \in \mathcal{L}(\mathbb{C}^d)$, $U \in \mathcal{U}_d$.

Prop Let $\rho_{AB}(x) := (1-x)\, \phi^+_{AB} + \frac{x}{d^2} \mathbb{1}_{AB}$ with $x \in \left[0, \frac{d^2}{d^2-1}\right]$.

i) ρ_{AB} is separable if and only if $x \geq \frac{d}{d+1}$.

ii) Let σ_{AB} be arbitrary with $\beta := \mathrm{tr}(\sigma_{AB} \phi^+_{AB}) = \langle \phi^+ | \sigma_{AB} | \phi^+ \rangle$

Then $\int_{\mathcal{U}_d} (U \otimes \bar{U})\, \sigma_{AB}\, (U \otimes \bar{U})^\dagger = \rho_{AB}(y)$

where $y = \frac{d^2}{d^2-1}(1-\beta)$.

Proof: See exercise sheet #6. $\square$