CHAPTER 3: SCHUR-WEYL DUALITY

§3.1 Representations of direct product groups

Def (Direct product of groups)

Let G and H be groups. The direct product $G \times H$ is the Cartesian set $G \times H = \{(g,h) : g \in G, h \in H\}$ with multiplication

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

for all $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Def (External product representations)

Let (φ, V) and (ψ, W) be representations of groups G and H, respectively. Then $V \otimes W$ affords the external product representation of the direct product $G \times H$ by defining

$$(\varphi \hat{\otimes} \psi)(g,h) := \varphi(g) \otimes \psi(h).$$

In order to distinguish this from the tensor representation in §2.1, we sometimes write $V \hat{\otimes} W$ for the representation space.

<u>Observations:</u> i) If (φ, V) and (ψ, W) are irreducible, then so is $(\varphi \hat{\otimes} \psi, V \hat{\otimes} W)$.

ii) Every irreducible representation of $G \times H$ arises this way.

§3.2 Double commutant theorem

Def (Commutant)

Let $\mathcal{A}$ be a subset of an algebra $\mathcal{C}$. The commutant $\mathcal{A}'$ of $\mathcal{A}$ is the set of elements in $\mathcal{C}$ commuting with all of $\mathcal{A}$:

$$\mathcal{A}' = \{ b \in \mathcal{C} : ab = ba \text{ for all } a \in \mathcal{A} \}.$$

For a vector space V we denote by $\mathrm{End}(V)$ the algebra of operators acting on V.

Lem Let V and W be finite-dimensional complex vector spaces. The commutant of $\mathrm{End}(V) \otimes \mathbb{1}_W$ in $\mathrm{End}(V \otimes W) \cong \mathrm{End}(V) \otimes \mathrm{End}(W)$ is $\mathbb{1}_V \otimes \mathrm{End}(W)$.

Proof: Set $\mathcal{A} = \mathrm{End}(V) \otimes \mathbb{1}_W$ and $\mathcal{B} = \mathbb{1}_V \otimes \mathrm{End}(W)$.

Clearly, an element $\mathbb{1}_V \otimes b \in \mathcal{B}$ commutes with every $a \otimes \mathbb{1}_W \in \mathcal{A}$, and hence $\mathcal{B} \subseteq \mathcal{A}'$.

Let now $a \otimes \mathbb{1}_W \in \mathcal{A}$ and $\tilde{a} \in \mathcal{A}'$ be arbitrary, and write

$$a \otimes \mathbb{1}_W = \begin{pmatrix} a & 0 & & 0 \\ 0 & a & & \\ & & \ddots & \\ 0 & & & a \end{pmatrix}, \quad \tilde{a} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & & \vdots \\ \vdots & & & \\ \tilde{a}_{n1} & \cdots & & \tilde{a}_{nn} \end{pmatrix}. \quad (n = \dim W)$$

Then we have

$$(a \otimes \mathbb{1}_W)\hat{a} = \begin{pmatrix} a\tilde{a}_{11} & a\tilde{a}_{12} & \cdots & a\tilde{a}_{1n} \\ a\hat{a}_{21} & a\hat{a}_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a\tilde{a}_{n1} & \cdots & & a\tilde{a}_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{a}_{11}a & \tilde{a}_{12}a & \cdots & \tilde{a}_{1n}a \\ \hat{a}_{21}a & \tilde{a}_{22}a & & \vdots \\ \vdots & & & \vdots \\ \tilde{a}_{n1}a & \cdots & & \tilde{a}_{nn}a \end{pmatrix} = \tilde{a}\,(a \otimes \mathbb{1}_W).$$

Hence, for fixed i,j we have $[a, \tilde{a}_{ij}] = 0$ for all $a \in End(V)$, and therefore $\tilde{a}_{ij} = \lambda_{ij} \mathbb{1}_A$ for some $\lambda_{ij} \in \mathbb{C}$. Let $b \in End(W)$ be defined by $(b)_{ij} = \lambda_{ij}$, then $\tilde{a} = \mathbb{1}_A \otimes b \in \mathbb{1}_A \otimes End(W) = \mathcal{B}$, and thus $\mathcal{A}' \subseteq \mathcal{B}$. $\square$

With this in hand we can prove the double commutant thm.

Prop (Double commutant theorem)

Let (φ, V) be a representation of a finite group G with decomposition $V = \bigoplus_\alpha V_\alpha \otimes \mathbb{C}^{n_\alpha}$ into pairwise inequivalent irreducible representations V_α with multiplicity n_α.

Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be the subalgebra generated by φ, and set $\mathcal{B} = \mathcal{A}'$. Then we have the following:

i) $\mathcal{A} \cong \bigoplus_\alpha \mathrm{End}(V_\alpha) \otimes \mathbb{1}_{\mathbb{C}^{n_\alpha}}$

ii) $\mathcal{B} \cong \bigoplus_\alpha \mathbb{1}_{V_\alpha} \otimes \mathrm{End}(\mathbb{C}^{n_\alpha})$

iii) $\mathcal{B}' = (\mathcal{A}')' = \mathcal{A}$

__Proof:__ Let $(\varphi_\alpha, V_\alpha)$ be the irreducible representations appearing in (φ, V), and set $d_\alpha = \dim V_\alpha$.

i) $\supseteq$: An application of Schur's lemma (Serre, Sec. 2.2) shows that

$$\mathcal{A} \ni d_\alpha \sum_{g \in G} \overline{\varphi_\alpha(g)_{ij}}\, \varphi(g) = E_{ij}^{(\alpha)} \otimes \mathbb{1}_{\mathbb{C}^{n_\alpha}},$$

where $\varphi_\alpha(g)_{ij}$ is the (i,j) matrix coefficient of $\varphi_\alpha(g)$, and $E_{ij}^{(\alpha)}$ is the (i,j)-elementary matrix in $\mathrm{End}(V_\alpha)$.

Since the $E_{ij}^{(\alpha)}$ are a basis of $\mathrm{End}(V_\alpha)$, we have that

$$\mathcal{A} \supseteq \bigoplus_\alpha \mathrm{End}(V_\alpha) \otimes \mathbb{1}_{\mathbb{C}^{n_\alpha}}.$$

$\subseteq$ holds by the decomposition of V into isotypical components $V_\alpha \otimes \mathbb{C}^{n_\alpha}$, and hence we have equality.

ii) $\subseteq$ Let P_α be the projection onto $V_\alpha \otimes \mathbb{C}^{n_\alpha}$, i.e.,

$$P_\alpha \mathcal{A} = V_\alpha \otimes \mathbb{C}^{n_\alpha}.$$

Then every $b \in \mathcal{B}$ commutes with P_α by definition, and hence $b = \mathbb{1}_{\mathcal{A}} b = \sum_\alpha P_\alpha b = \sum_\alpha \underbrace{P_\alpha b P_\alpha}_{=: b_\alpha} = \sum_\alpha b_\alpha$

where $b_\alpha \in \mathrm{End}(V_\alpha \otimes \mathbb{C}^{n_\alpha})$. By the preceding lemma,

$b_\alpha = \mathbb{1}_{V_\alpha} \otimes \tilde{b}_\alpha$ for some $\tilde{b}_\alpha \in \mathrm{End}(\mathbb{C}^{n_\alpha})$.

$\supseteq$ clearly holds since $\bigoplus_\alpha \mathbb{1}_{V_\alpha} \otimes b_\alpha$ for $b_\alpha \in \mathrm{End}(\mathbb{C}^{n_\alpha})$ commutes with any $\bigoplus_\alpha a_\alpha \otimes \mathbb{1}_{\mathbb{C}^{n_\alpha}} \in \mathcal{A}$.

iii) Follows similarly to ii). $\square$

§ 3.3 Schur-Weyl duality

We now focus on the following two groups:

- the symmetric group $S_n = \{ f : \{1,\dots,n\} \to \{1,\dots,n\} \text{ bijective} \}$
- the unitary group $\mathcal{U}_d = \{ U \in \mathcal{L}(\mathbb{C}^d) : U^\dagger U = U U^\dagger = \mathbb{1}_d \}$.

The two groups have the following representations on $(\mathbb{C}^d)^{\otimes n}$:

$$\pi \in S_n : \quad \varphi(\pi)\big(|\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle \big)$$
$$= |\psi_{\pi^{-1}(1)}\rangle \otimes |\psi_{\pi^{-1}(2)}\rangle \otimes \dots \otimes |\psi_{\pi^{-1}(n)}\rangle$$

$$U \in \mathcal{U}_d : \quad \omega(U)\big(|\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle \big)$$
$$= U|\psi_1\rangle \otimes \dots \otimes U|\psi_n\rangle$$

(+ linear extension)

Def (Symmetric subspace)

The symmetric subspace $\mathrm{Sym}^n(V)$, also called n-th symmetric power of V, is the subspace

$$\mathrm{Sym}^n(V) = (V^{\otimes n})^{S_n} = \{ |v\rangle \in V^{\otimes n} : \varphi(\pi)|v\rangle = |v\rangle \text{ for all } \pi \in S_n \}.$$

With $P = \frac{1}{n!} \sum_{\pi \in S_n} \varphi(\pi)$, we have $\mathrm{Sym}^n(V) = P\, V^{\otimes n}$.

Lem $\mathrm{Sym}^n(V) = \mathrm{span}\{ |v\rangle^{\otimes n} : |v\rangle \in V \}$.

<u>Proof</u>: Let $\{|e_i\rangle\}_{i=1}^{d}$ be an ONB for V ($d = \dim V$).

By definition, $\mathrm{Sym}^n(V)$ is spanned by the vectors

$$|v_{i_1 \dots i_n}\rangle := \sum_{\pi \in S_n} \varphi(\pi)\left(|e_{i_1}\rangle \otimes \dots \otimes |e_{i_n}\rangle\right)$$

$$= \sum_{\pi \in S_n} |e_{i_{\pi^{-1}(1)}}\rangle \otimes \dots \otimes |e_{i_{\pi^{-1}(n)}}\rangle$$

for indices $i_j \in \{1, \dots, d\}$, $j = 1, \dots, n$.

We have $\mathrm{span}\{|v\rangle^{\otimes n} : |v\rangle \in V\} \subseteq \mathrm{Sym}^n(V)$.

To show the other inclusion, we rewrite the vectors $|v_{i_1 \dots i_n}\rangle$ using derivatives,

$$|v_{i_1 \dots i_n}\rangle = \partial_{\lambda_2} \dots \partial_{\lambda_n} \left(|e_{i_1}\rangle + \sum_{j=2}^{n} \lambda_j |e_{i_j}\rangle\right)^{\otimes n} \Big|_{\lambda_2 = \dots = \lambda_n = 0}.$$

Since by the definition of the derivative we have

$$\partial_{\lambda_j} \left(|e_1\rangle + \lambda_j |e_j\rangle\right)^{\otimes n} \Big|_{\lambda_j = 0} = \lim_{\lambda_j \to 0} \frac{\left(|e_1\rangle + \lambda_j |e_j\rangle\right)^{\otimes n} - |e_1\rangle^{\otimes n}}{\lambda_j},$$

the $|v_{i_1 \dots i_n}\rangle$ are limits of elements in $W = \mathrm{span}\{|v\rangle^{\otimes n} : |v\rangle \in V\}$, and since W is finite-dim. and hence closed, $|v_{i_1 \dots i_n}\rangle \in W$ for all sets of indices $i_1, \dots, i_n$. It follows that $\mathrm{Sym}^n(V) \subseteq W$. $\square$

$\boxed{\text{Cor}}$ Let $C \in \mathrm{End}(V^{\otimes n})$ be such that

$$\varphi(\pi)\, C\, \varphi(\pi)^{\dagger} = C \quad \text{for all } \pi \in S_n.$$

Then $C \in \mathrm{span}\{X^{\otimes n} : X \in \mathrm{End}(V)\}$

<u>Proof:</u> Let $W = \mathrm{End}(V^{\otimes n}) \cong \mathrm{End}(V)^{\otimes n}$, and for a fixed basis $\{|e_i\rangle\}_{i=1}^{d}$ $(d = \dim V)$ of V consider the basis $\{E_{ij}\}_{i,j=1}^{d}$ of $\mathrm{End}(V)$, where $E_{ij} : |e_k\rangle \mapsto \delta_{jk} |e_i\rangle$.

Denote by $\varphi : S_n \to GL(V^{\otimes n})$ the tensor representation of S_n on $V^{\otimes n}$, and by $\tilde{\varphi} : S_n \to GL(W)$ the analogous tensor rep. of S_n on $W = \mathrm{End}(V)^{\otimes n}$. Then $\tilde{\varphi}(\pi)$ acting on $X \in \mathrm{End}(V^{\otimes n})$ has the matrix representation $\varphi(\pi)\, X\, \varphi(\pi)^{-1}$.

The claim now follows from the lemma applied to $(\tilde{\varphi}, W)$. $\square$

In the following we will view $\omega : X \mapsto X^{\otimes n}$ as a representation of $GL(V) = \{X \in \mathrm{End}(V) : X \text{ is invertible}\}$.

$\boxed{\text{Prop}}$ A representation of $\mathcal{U}_d$ $(d = \dim V)$ is irreducible if and only if the corresponding rep. of $GL(V)$ is.

For a proof, see e.g. lecture notes by J. Alcock-Zeilinger.

Prop S_n and $GL(V)$ span each other's commutants in $\mathrm{End}(V^{\otimes n})$.

<u>Proof</u>: Let $\mathcal{A} \subseteq \mathrm{End}(V^{\otimes n})$ be the subalgebra generated by $\varphi(\pi)$, $\pi \in S_n$, and let $\mathcal{B} \subseteq \mathrm{End}(V^{\otimes n})$ be the subalgebra generated by $w(g)$, $g \in GL(V)$.

Since $\varphi(\pi)$ and $w(U)$ commute for all $\pi \in S_n$, $U \in \mathcal{U}_d$, we clearly have $\mathcal{B} \subseteq \mathcal{A}'$.

The previous corollary shows that $\mathcal{A}' = \mathrm{span}\{X^{\otimes n} : X \in \mathrm{End}(V)\}$.

Let $X \in \mathrm{End}(V)$, then $X + t\mathbb{1}$ is invertible for all but finitely many t, and so $(X+t\mathbb{1})^{\otimes n} \in \mathcal{B}$ for all but finitely many t.

But $(X+t\mathbb{1})^{\otimes n}$ is a polynomial in t of degree n, and by Lagrange's interpolation theorem determined by any $n+1$ distinct points.

Hence, $(X+t\mathbb{1})^{\otimes n} \in \mathcal{B}$ for all t, in particular for $t=0$.

It follows that $\mathcal{A}' = \mathrm{span}\{X^{\otimes n} : X \in \mathrm{End}(V)\} \subseteq \mathcal{B}$, hence $\mathcal{A}' = \mathcal{B}$.

The fact that $\mathcal{B}' = \mathcal{A}$ now follows from the Double Commutant Theorem, concluding the proof. $\square$

Prop (Schur-Weyl duality)

Let $V = \mathbb{C}^d$ and $(\varphi, V^{\otimes n})$ and $(\omega, V^{\otimes n})$ be the tensor representations of S_n and $GL(V)$ defined above.

As a representation of $S_n \times GL(V)$, the space $V^{\otimes n}$ decomposes as

$$V^{\otimes n} = \bigoplus_{\lambda} V_\lambda \otimes U_\lambda,$$

where $(\varphi_\lambda, V_\lambda)$ and $(\omega_\lambda, U_\lambda)$ are inequivalent irreducible representations of S_n and $GL(V)$, respectively, and

$$\varphi(\pi) = \bigoplus_{\lambda} \varphi_\lambda(\pi) \otimes \mathbb{1}_{U_\lambda}, \quad \pi \in S_n$$

$$\omega(g) = \bigoplus_{\lambda} \mathbb{1}_{V_\lambda} \otimes \omega_\lambda(g). , \quad g \in GL(V)$$

The same assertion holds when replacing $GL(V)$ with $\mathcal{U}_d$.

<u>Proof:</u> The decomposition of $V^{\otimes n}$ follows from the Double Commutant Theorem and the fact that S_n and $GL(V)$ span each other's commutant.

It remains to show that $U_\lambda \cong \hom_{S_n}(V_\lambda, V^{\otimes n})$ is an irreducible representation of $GL(V)$ (or $\mathcal{U}_d$).

By Schur's lemma, this is equivalent to showing that

$$\mathrm{End}_{GL(V)}(U_\lambda) := \hom_{GL(V)}(U_\lambda, U_\lambda) \cong \mathbb{C}.$$

We have $Z(\mathrm{End}(U_\lambda)) \cong \mathbb{C}$ (Z... center of an algebra).

Schur's lemma and the above decomposition show that

$$\mathrm{End}_{S_n}(V^{\otimes n}) \cong \bigoplus_\lambda \mathrm{End}(U_\lambda)$$

$$\mathrm{End}_{GL(V)\times S_n}(V^{\otimes n}) \cong \bigoplus_\lambda \mathrm{End}_{GL(V)}(U_\lambda).$$

Since $\mathrm{End}_{S_n}(V^{\otimes n}) = \mathrm{span}\{X^{\otimes n} : X \in GL(V)\}$, we have

$$\mathrm{End}_{GL(V)\times S_n}(V^{\otimes n}) \subseteq Z(\mathrm{End}_{S_n}(V^{\otimes n})),$$

and hence also $\mathrm{End}_{GL(V)}(V^{\otimes n}) \subseteq Z(\mathrm{End}(U_\lambda)) \cong \mathbb{C}$. $\square$

<u>Summary:</u> Schur-Weyl duality says that

$$V^{\otimes n} \cong \bigoplus_\lambda V_\lambda \otimes U_\lambda$$

as a representation of $S_n \times U_d$, with V_λ and U_λ irreps of S_n and U_d, respectively.

Next chapter: Discussion of the index λ and the irreps V_λ, U_λ.