CHAPTER 2: BASICS FROM REPRESENTATION THEORY CONT.

§2.1 Tensor and dual representations, hom spaces

Tensor representation: Let (φ, V) and (ψ, W) be rep's of a group G. Then $(\varphi \otimes \psi)(g) := \varphi(g) \otimes \psi(g)$ defines a representation on $V \otimes W$ called the tensor representation.

<u>Note:</u> Even for V and W irreducible the representation $V \otimes W$ is in general <u>reducible</u>.

<u>Example:</u> Let (φ, V) be a rep of G and consider the tensor rep $W = V \otimes V$. Let $\mathbb{F}: V \otimes V \to V \otimes V$ be the swap operator, defined by $\mathbb{F}(|x\rangle \otimes |y\rangle) = |y\rangle \otimes |x\rangle$ for $|x\rangle, |y\rangle \in V$ + lin. ext.

Then $V \otimes V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$, where

$$\mathrm{Sym}^2(V) = \{|z\rangle \in V \otimes V : \mathbb{F}|z\rangle = |z\rangle\}$$
$$\mathrm{Alt}^2(V) = \{|z\rangle \in V \otimes V : \mathbb{F}|z\rangle = -|z\rangle\}$$

$\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ are each G-invariant subspaces called the symmetric and antisymmetric square, respectively. (also called (anti-) symmetric subspace.)

Def (Dual representation)

Let (φ, V) be a representation of G. The dual representation (φ^*, V^*) is defined for $g \in G$ and $L \in V^* := \{ f: V \to \mathbb{C} \text{ linear} \}$ as
$$\varphi^*(g)(L) := L \circ \varphi(g)^{-1}.$$

The dual representation satisfies for all $g \in G$, $|v\rangle \in V$, $\langle w| \in V^*$:

i) $\varphi^*(g) = \varphi(g^{-1})^T$

ii) $\left(\varphi^*(g)\langle w|\right)\left(\varphi(g)|v\rangle\right) = \langle w|\varphi^*(g)^T \varphi(g)|v\rangle = \langle w|v\rangle$

Observations: ∙) (φ^*, V^*) is irreducible iff (φ, V) is.

∙) If (φ, V) is unitary, then $\varphi^*(g) = \overline{\varphi(g)}$ (complex conjugate).

Def ∙) Let V, W be two vector spaces over the same field $\mathbb{F}$.

Then $\hom(V, W) := \{ f: V \to W \text{ linear} \}$ is the vector space of linear maps from V to W over $\mathbb{F}$.

∙) Let (φ, V) and (ψ, W) be rep's of a group G. Then G acts on $\hom(V, W)$ by sending $f: V \to W$ to
$$\psi(g) \circ f \circ \varphi(g)^{-1},$$
which turns $\hom(V, W)$ into a representation of G.

Some observations:

1) Setting $W = \mathbb{C}$ with the trivial action of G, we recover the dual representation from this construction.

2) $\hom(V, W) \cong V^* \otimes W$ as vector spaces and representations.

3) Let $V^G := \{ |v\rangle \in V : \varphi(g)|v\rangle = |v\rangle \text{ for all } g \in G \}$

Then $\hom_G(V, W) := \hom(V, W)^G$

$$= \{ f : V \to W : \psi(g) \circ f \circ \varphi(g^{-1}) = f \ \forall g \in G \}$$

$$= (V^* \otimes W)^G$$

4) Let $V = \bigoplus_i V_i$ be an isotypical decomposition with isotypical components $V_i = W_i^{n_i}$ for pairwise inequivalent irreducible representations W_i of G. Then

$$n_i = \dim \hom_G(V, V_i) = \dim (V^* \otimes V_i)^G.$$

§2.2 Group algebra and characters

Recall the regular representation of a finite group G with $n = |G|$. Let $V \cong \mathbb{C}^n$ be the n-dimensional vector space with basis $\{|g\rangle\}_{g \in G}$.

That is, V consists of formal linear combinations of elements in G.

The group multiplication endows V with the structure of an algebra:

$$\Big(\sum_{g\in G} c_g |g\rangle\Big)\cdot\Big(\sum_{h\in G} d_h |h\rangle\Big) = \sum_{g,h\in G} c_g d_h |gh\rangle$$

$$= \sum_{g\in G} f_g |g\rangle$$

where $f_g = \sum_{h\in G} c_{gh^{-1}} d_h$.

This multiplication satisfies:

- Associativity;
- the group identity element e is the multiplicative identity;
- Distributivity w.r.t. addition.

Hence, $(V, +, \cdot)$ has the structure of an algebra. It is called the group algebra $\mathbb{C}[G]$ (or $\mathbb{C}G$ or $\mathcal{A}_{\mathbb{C}}(G)$).

A representation of an algebra $\mathcal{A}$ over a field $\mathbb{F}$ is an algebra homomorphism $\mathcal{A} \to \mathrm{End}_{\mathbb{F}}(V)$ into the algebra of endomorphisms on an $\mathbb{F}$-vector space V (with multiplication given by composition of linear operators on V).

For $\mathcal{A} = \mathbb{C}[G]$, any rep (φ, V) of G can be extended to a rep $(\tilde{\varphi}, V)$ of $\mathbb{C}[G]$ on V by setting $\tilde{\varphi}(|g\rangle) = \varphi(g)$ + linear extension.

Conversely, any rep $(\tilde{\varphi}, V)$ of $\mathbb{C}[G]$ yields a representation of G by restricting $\tilde{\varphi}$ to $\{1g\}$.

$\Rightarrow$ representations of G correspond exactly to rep's of $\mathbb{C}[G]$.

Elements in $\mathbb{C}[G]$ can be interpreted as functions $G \to \mathbb{C}$.

There is a special set of functions called class functions:

$$f(g) = f(hgh^{-1}) \quad \text{for all } g,h \in G,$$

i.e., functions that are constant on the conjugacy classes of G.

The set of class functions is identical to the center

$$Z(\mathbb{C}[G]) = \{ f \in \mathbb{C}[G] : fg = gf \ \forall g \in \mathbb{C}[G] \}$$

$\boxed{\text{Def}}$ (Character of a representation)

Let (φ, V) be a representation of G. The character $\chi = \chi_V$ of (φ, V) is the class function defined as

$$\chi(g) = \operatorname{tr} \varphi(g).$$

<u>Properties:</u> i) $\dim V = \chi(e)$

ii) (φ, V) unitary: $\chi(g^{-1}) = \overline{\chi(g)}$

iii) $\chi_{V \oplus W} = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

For $x = \sum_{g \in G} x_g |g\rangle \in \mathbb{C}[G]$ and $y = \sum_{g \in G} y_g |g\rangle \in \mathbb{C}[G]$

we set $(x,y) := \frac{1}{|G|} \sum_{g \in G} \overline{x}_g y_g$, the canonical inner product on $\mathbb{C}[G]$.

$\boxed{\text{Prop}}$ Let W_i, $i=1,\dots,k$ be pairwise inequivalent irreducible representations of a group G, and denote by χ_i the corresponding characters. Then $(\chi_i, \chi_j) = \delta_{ij}$.

Moreover, any class function orthogonal to all χ_i is identically 0.

Hence, $\{\chi_i\}_{i=1}^{k}$ is an orthonormal basis of the set of class functions.

<u>Proof</u>: Serre's book, Teleman's notes. □

$\boxed{\text{Cor}}$ i) The multiplicity of an irreducible rep W in some rep V is (χ_V, χ_W).

ii) V is irreducible iff $(\chi_V, \chi_V) = 1$.

iii) Two representations are isomorphic iff they have the same character.

iv) The number of distinct irreducible representations of a finite group G is equal to the number of conjugacy classes.

Character of the regular representation $R(G)$:

Recall that $R(G)$ has basis $\{|g\rangle\}_{g\in G}$ and

G acts by left multiplication: $\varphi(g): |h\rangle \mapsto |gh\rangle$

$$\Rightarrow \quad \chi_{R(G)}(g) = \operatorname{tr}\varphi(g) = |G|\,\delta_{g,e}$$

$\boxed{\text{Cor}}$ i) The multiplicity of any irreducible representation in the regular representation is equal to its dimension.

ii) Let $W_1, \dots, W_k$ be a complete list of irreducible representations of G. Then every W_i appears in $R(G)$, and

$$\sum_{i=1}^{k} (\dim W_i)^2 = |G|.$$

$\boxed{\text{Prop}}$ Let (φ, V) be a representation of G and W be a fixed irreducible representation with character χ_W. Then the projection onto the isotypical component of W in V is given by

$$P_W = \frac{\dim W}{|G|} \sum_{g\in G} \overline{\chi_W(g)}\, \varphi(g).$$

In particular, $P = \frac{1}{|G|} \sum_{g\in G} \varphi(g)$ projects onto

$$V^G = \{|v\rangle \in V : \varphi(g)|v\rangle = |v\rangle \text{ for all } g \in G\}.$$

§2.3 From finite to compact groups

Def (Topological and compact groups)

A topological group is a group G endowed with a topology s.t. group multiplication and inversion are continuous.

A compact group is a topological group that is compact, i.e., every open cover of G has a finite subcover. Closed subgroups of compact groups are compact as well.

Def (Representation of a topological group)

A representation (φ, V) of a topological group G on a (normed, finite-dim.) vector space V is a continuous group homomorphism

$$\varphi : G \to GL(V).$$

Recall from §1.1-2.2 that the averaging operation $\frac{1}{|G|}\sum_{g\in G}$ over a finite group was essential for proving Maschke's theorem, the character formulas, etc.

For compact groups we can replace this discrete averaging by a suitable integral to recover many of the previous results for finite groups also for compact groups.

Prop (Haar measure)

Let G be a compact group. There exists a unique measure dg on G, called the Haar measure, satisfying:

i) Invariance: For every continuous function $f: G \to \mathbb{C}$ and every $h \in G$, $\int_G f(g)\,dg = \int_G f(gh)\,dg = \int_G f(hg)\,dg$.

ii) Normalization: $\int_G 1\,dg = 1$.

Examples: 1. Every finite group is a compact group (w.r.t. the discrete topology), and $dg = \frac{1}{|G|}$, $\int_G dg = \frac{1}{|G|} \sum_{g \in G}$.

2. The circle group $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} = \{ \exp(i\theta) : \theta \in [0, 2\pi \}$ has Haar measure $dg = \frac{1}{2\pi}\,d\theta$.

Using the Haar measure, one can prove analogous statements about finite-dimensional representations of compact groups, e.g.:

i) Every G-invariant subspace has a G-invariant complement.

ii) Every representation decomposes as a sum of irreducible reps.

iii) Most aspects of character theory (careful about $|G|$)

The regular representation of a compact group G is defined as the Hilbert space $L^2(G)$ of square-integrable functions on G, with the G-action given by $\varphi(g) f(h) = f(g^{-1}h)$.

If $|G| = \infty$ then also $\dim L^2(G) = \infty$. However, we have:

$\boxed{\text{Prop}}$ (Peter-Weyl theorem)

Let G be a compact group.

i) The linear span of all matrix coefficients of the irreducible unitary representations of G is dense in $L^2(G)$.

ii) Every irreducible unitary representation of G is finite-dim.

iii) The (infinite-dim!) regular representation $L^2(G)$ decomposes into a direct sum of the irreducible unitary representations of G, each one occurring with multiplicity equal to its dimension.

The matrix coefficients of the complete set of irreps form an orthonormal basis of $L^2(G)$.

<u>Proof</u>: Knapp, Thm. 1.12. $\square$