CHAPTER 10: QUANTUM STATE TOMOGRAPHY

§ 10.1 Warm-up: pure state estimation

Quantum state tomography is the task of obtaining a classical description of an unknown quantum state ρ.

Same assumptions as in spectrum estimation:

1) We can prepare identical copies of the unknown state ρ.
2) We can make joint measurements on all copies simultaneously.

→ Task: Find measurement on $\rho^{\otimes n}$ that yields asymptotically accurate estimate of ρ (as $n \to \infty$).

As a warm-up, consider the special case of pure state estimation:

Assuming a quantum system $\mathcal{H}$ is prepared in unknown pure state $|\psi\rangle$, estimate ψ by measuring $|\psi\rangle\langle\psi|^{\otimes n}$.

We know from previous chapters:

a) $\psi^{\otimes n}$ is permutation-invariant, and moreover,

b) $\psi^{\otimes n}$ is supported on the symmetric subspace!

Ansatz for the measurement:

$$\left\{ \binom{n+d-1}{n} |\varphi\rangle\langle\varphi|^{\otimes n} \right\}_{|\varphi\rangle \in \mathcal{H}},$$

where $|\varphi\rangle$ is distributed according to the Haar measure $d\varphi$ on pure states (cf. Chapter 7: The de Finetti thm.).

This is a continuous POVM on $\mathrm{Sym}^n(\mathcal{H})$:

- $|\varphi\rangle\langle\varphi|^{\otimes n} \geq 0 \quad \forall |\varphi\rangle \in \mathcal{H}$
- $\binom{n+d-1}{n} \int d\varphi \, |\varphi\rangle\langle\varphi|^{\otimes n}$ is the projector onto $\mathrm{Sym}^n(\mathcal{H})$, and hence equal to the identity on that space.

Remark: Recall from Chapter 5 that projective measurements correspond to the spectral decomposition of Hermitian observables.

A POVM can be "purified" to a projective measurement on a larger space (= system + environment).

How can we then make sense of a continuous POVM on a finite-dimensional Hilbert space?

Answer by Chiribella et al (arXiv: quant-ph/0702068)

A continuous POVM can be expressed as a continuous random variable taking values in some set Ω with probability density p_Ω. For each $w \in \Omega$ there is a discrete, finite POVM M_w.

The outcome of the continuous POVM is obtained by

1) sample $w \in \Omega$ according to p_Ω.
2) Measure system with M_w.

Back to pure state estimation!

We propose the following protocol:

1) Measure $|\psi\rangle\langle\psi|^{\otimes n}$ w.r.t. $\{Q_\varphi\}_{|\varphi\rangle \in \mathcal{X}}$,

where $Q_\varphi := \binom{n+d-1}{n} |\varphi\rangle\langle\varphi|^{\otimes n}$.

2) Outcome $|\hat{\psi}\rangle$ is our estimator for $|\psi\rangle$.

Claim: $\mathbb{E}_{\hat{\psi}}\left(F(\hat{\psi},\psi)^2\right) = \mathbb{E}_{\hat{\psi}}\left(|\langle\hat{\psi}|\psi\rangle|^2\right) \geq 1 - \frac{d}{n}$

$\Rightarrow \mathbb{E}_{\hat{\psi}} D(\hat{\psi},\psi) \leq \sqrt{\frac{d}{n}} \longrightarrow 0$ as $n \to \infty$.

Proof: Exercise! (We essentially calculated this in Ch. 7)

§ 10.2 Symmetries of state tomography

Unknown quantum state: $\rho = \sum_i \underset{\substack{\uparrow \\ \text{spectrum}}}{r_i} \; |e_i\rangle\langle e_i|$ ← eigenbasis

Pure state estimation:

POVM operators $\propto |\psi\rangle\langle\psi|^{\otimes n} = (U|\psi_0\rangle\langle\psi_0|U^\dagger)^{\otimes n}$

where $|\psi_0\rangle$ is a fixed pure state

(with spectrum $(1, 0, \dots, 0)$),

and U is a random unitary.

Idea for full tomography: replace $|\psi_0\rangle\langle\psi_0|$ above by a diagonal state with an estimate of the spectrum, and take Haar-random U as estimate for the eigenbasis!

The guess for $\mathrm{spec}(\rho)$ is obtained as in spectrum estimation: Measure $\rho^{\otimes n}$ with respect to Schur-Weyl decomposition, $\{P_\lambda\}_{\lambda \vdash n}$ where P_λ projects onto λ-comp. in $(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash n} V_\lambda \otimes W_\lambda$.

Remains to incorporate U in the measurement:

Denote by $\mathcal{D}(\mathcal{H}) = \{ \rho \in \mathcal{L}(\mathcal{H}) : \rho \geq 0, \operatorname{tr} \rho = 1 \}$ the set of density operators on $\mathcal{H}$.

We look for a continuous POVM $\{ M_\sigma \}_{\sigma \in \mathcal{D}(\mathcal{H})}$ with the following symmetries:

1) Permutation invariance: $Q_\pi M_\sigma Q_\pi^\dagger = M_\sigma \quad \forall \pi \in S_n, \sigma \in \mathcal{D}(\mathcal{H})$

(since $\rho^{\otimes n}$ has this symmetry)

2) Unitary covariance:

$$M_{U\sigma U^\dagger} = U^{\otimes n} M_\sigma (U^\dagger)^{\otimes n} \qquad \forall U \in \mathcal{U}_d, \sigma \in \mathcal{D}(\mathcal{H})$$

(since we consider ρ and $U\rho U^\dagger$ equally likely, and hence

$$\operatorname{tr}(\rho^{\otimes n} M_\sigma) = \operatorname{tr}\left(U^{\otimes n} \rho^{\otimes n} (U^\dagger)^{\otimes n} U^{\otimes n} M_\sigma (U^\dagger)^{\otimes n} \right)$$
$$\stackrel{!}{=} \operatorname{tr}\left((U\rho U^\dagger)^{\otimes n} M_{U\sigma U^\dagger} \right))$$

Ansatz satisfying both 1) and 2):

Let $\lambda \vdash_d n$ be a Young diagram. Set $\bar{\lambda} = \frac{1}{n} \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ (diagonal quantum state), and for $U \in \mathcal{U}_d$ set

$$\sigma(\lambda, U) := U \bar{\lambda} U^\dagger .$$

Define measurement operators for $\sigma = \sigma(\lambda, U)$

$$M_\sigma \equiv M(\lambda, U) = c_\lambda \; P_\lambda \, (U \bar{\lambda} U^\dagger)^{\otimes n} P_\lambda .$$

On outcome (λ, U) we take the estimator $\sigma = U \bar{\lambda} U^\dagger$.

What is the constant c_λ ?

Observe that the operator $M_\lambda := \int dU \, M(\lambda, U)$ satisfies:

- $Q_\pi M_\lambda Q_\pi^\dagger = M_\lambda \quad \forall \pi \in S_n$
- $U^{\otimes n} M_\lambda (U^\dagger)^{\otimes n} = M_\lambda \quad \forall U \in \mathcal{U}_d$
- $M_\lambda = P_\lambda M_\lambda P_\lambda \in \mathrm{End}(V_\lambda \otimes W_\lambda)$

(recall: Schur-Weyl duality: $(\mathbb{C}^d)^{\otimes n} = \bigoplus_{\lambda \vdash_d n} \overset{\dim = d_\lambda}{V_\lambda} \otimes \underset{\dim = m_\lambda}{W_\lambda}$)

$\Rightarrow$ want $M_\lambda = P_\lambda$, since then

$$\mathbb{1} = \sum_\lambda P_\lambda = \sum_\lambda M_\lambda = \sum_\lambda \int dU \, M(\lambda, U) .$$

$\Rightarrow$ compute c_λ by taking traces:

$$\mathrm{tr}\, M_\lambda = c_\lambda \int dU \, \mathrm{tr}\big[\underbrace{P_\lambda (U \bar{\lambda} U^\dagger)^{\otimes n} P_\lambda}\big]$$

$= \mathbb{1}_{V_\lambda} \otimes w_\lambda(U \bar{\lambda} U^\dagger)$, where w_λ is the irrep of $GL(\mathcal{X})$ on $\mathcal{X}^{\otimes n}$ labeled by λ.

Hence,

$$\mathrm{tr}\left(P_\lambda\,(U\bar{\lambda}U^\dagger)^{\otimes n}P_\lambda\right) = \mathrm{tr}\left[\mathbb{1}_{V_\lambda}\otimes w_\lambda(U\bar{\lambda}U^\dagger)\right]$$
$$= d_\lambda\cdot\mathrm{tr}\left(w_\lambda(U\bar{\lambda}U^\dagger)\right)$$
$$\equiv d_\lambda\cdot s_\lambda(U\bar{\lambda}U^\dagger)$$

where the Schur polynomial s_λ is the character of the irreducible representation (w_λ, W_λ) of $GL(\mathcal{H})$ (or $\mathcal{U}_d$) on $\mathcal{H}^{\otimes n}$ labeled by $\lambda\vdash_d n$.

Since characters are functions of eigenvalues (as traces), we have $s_\lambda(U\bar{\lambda}U^\dagger) = s_\lambda(\bar{\lambda})$, and hence,

$$\begin{aligned}\mathrm{tr}\,P_\lambda &= \dim(V_\lambda\otimes W_\lambda) = d_\lambda m_\lambda\\ \parallel\\ \mathrm{tr}\,M_\lambda &= c_\lambda\int dU\,\mathrm{tr}\left(P_\lambda(U\bar{\lambda}U^\dagger)^{\otimes n}P_\lambda\right)\\ &= c_\lambda d_\lambda\int dU\,s(\bar{\lambda})\\ &= c_\lambda d_\lambda\,s(\bar{\lambda})\end{aligned}$$

$$\Rightarrow\quad c_\lambda = \frac{m_\lambda}{s(\bar{\lambda})}$$

§10.3 Error analysis of our tomography protocol

We will need the following bounds on Schur polynomials:

$\boxed{\text{Lem}}$ Let $\lambda \vdash_d n$.

i) For $\bar{\lambda} = \frac{1}{n}\,\mathrm{diag}(\lambda)$, $\quad s_\lambda(\bar{\lambda}) \geq e^{-nH(\bar{\lambda})}$,

where $H(x) = \sum_{i=1}^{d} -x_i \log x_i$ is the Shannon entropy of a probability distribution $x = (x_1, \dots, x_d)$.

ii) Let $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ with $F(\rho,\sigma) = F$. Assume that $\mathrm{rk}(\rho) = r \leq d$. Then,

$$s_\lambda(\rho\sigma) \begin{cases} = 0 & \text{if } \lambda_{r+1} > 0; \\ \leq m_\lambda\, e^{-2nH(\bar{\lambda})} F^{2n} & \text{otherwise.} \end{cases}$$

Proof: See Haah et al., arXiv: 1508.01797.

We also recall the bound

$$d_\lambda = \dim V_\lambda \leq e^{nH(\bar{\lambda})},$$

which we implicitly used in the spectrum estimation chapter.

With this, we can estimate the probabilities of our continuous tomography POVM $\{M(\lambda, U)\}_{\lambda \vdash n,\, U \in \mathcal{U}_d}$:

$$\operatorname{tr}\left[M(\lambda,U)\,\rho^{\otimes n}\right] = \frac{m_\lambda}{s_\lambda(\bar\lambda)} \operatorname{tr}\left[P_\lambda\,(U\bar\lambda U^\dagger)^{\otimes n} P_\lambda\,\rho^{\otimes n}\right]$$

$$= \frac{m_\lambda}{s_\lambda(\bar\lambda)} \operatorname{tr}\Big[\underbrace{P_\lambda\,(U\bar\lambda U^\dagger)^{\otimes n} P_\lambda}_{=\mathbb{1}_{V_\lambda}\otimes w_\lambda(U\bar\lambda U^\dagger)}\;\underbrace{P_\lambda\,\rho^{\otimes n} P_\lambda}_{=\mathbb{1}_{V_\lambda}\otimes w_\lambda(\rho)}\Big]$$

$$= \frac{m_\lambda d_\lambda}{s_\lambda(\bar\lambda)} \operatorname{tr}\Big(\underbrace{w_\lambda(U\bar\lambda U^\dagger)\,w_\lambda(\rho)}_{=w_\lambda(U\bar\lambda U^\dagger\rho)}\Big)$$

$$= \frac{m_\lambda d_\lambda}{s_\lambda(\bar\lambda)}\, s_\lambda(U\bar\lambda U^\dagger\rho)$$

$$\leq m_\lambda\, e^{2nH(\bar\lambda)}\, m_\lambda\, e^{-2nH(\bar\lambda)}\, F^{2n}$$

$$\leq m_\lambda^2\, F^{2n}$$

$$\leq (n+1)^{2dr}\, F^{2n},$$

where we used $m_\lambda \leq (n+1)^{dr}$ for λ with $\lambda_h = 0$ for $h \geq r+1$,

and we set $F = F(\rho, U\bar\lambda U^\dagger)$.

$\Rightarrow$ for all $\varepsilon > 0$, with $\hat\rho = U\bar\lambda U^\dagger$,

$$\Pr\left(F(\rho,\hat\rho) \leq 1-\varepsilon\right) \leq (n+1)^{2dr}\,(1-\varepsilon)^{2n}.$$

more details on this proof: see arXiv: 1508.01797