CHAPTER 1: BASICS FROM REPRESENTATION THEORY

§1.1 Representations

Def (Group)

A group $(G, \cdot)$ is a set G together with a binary operation $\cdot : G \times G \to G$ satisfying:

i) Associativity: $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ for all $a, b, c \in G$.

ii) Identity element: there is $e \in G$ with $eg = ge = g$ for all $g \in G$.

iii) Inverse: for all $g \in G$ there exists $h \in G$ with $gh = hg = e$.

This inverse is unique and denoted g^{-1}.

__Examples:__ $(\mathbb{F}, +)$ for a field $\mathbb{F}$, $S_n = \{\text{bijections on } \{1, \dots, n\}\}$,

$GL(V) = \{\text{invertible linear maps } V \to V\}$ for vector space V.

Representation theory: study groups using linear algebra by letting them "act" on vector spaces.

Action of a group G on a set X:

map $\varphi : G \times X \to X$ satisfying for all $x \in X$ and $g, h \in G$ that

i) $\varphi(e, x) = x$,

ii) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$.

Def (Representation)

A (linear) representation (φ, V) of a group G on a vector space V (over a field $\mathbb{F}$) is a group homomorphism

$$\varphi : G \to GL(V),$$

that is, a map φ satisfying $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in G$.

A representation always satisfies $\varphi(e) = \mathbb{1}_V$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$.

The dimension or degree of the representation (φ, V) is the dimension of V. We will only deal with finite-dimensional representations in this course.

Examples: Let $G = \{e, g, g^2, \dots, g^{d-1}\}$ be a cyclic group of order d with (abstract) generator g satisfying $g^d = e$.
(as a concrete example, take $G = \mathbb{Z}/d\mathbb{Z}$).

Let $V = \mathbb{C}^d$ with basis $|0\rangle, |1\rangle, \dots, |d-1\rangle$, and define an operator $X \in \mathcal{L}(V)$ via $X|i\rangle = |i+1 \bmod d\rangle \ \forall i$.

Then the mapping $g \mapsto X$ determines a representation (φ, V) of G.

Different choice: φ' defined via $g \mapsto Z$, where

$Z|j\rangle = \omega^j |j\rangle$ for a (primitive) d-th root of unity.

These two representations are essentially the same:

$\boxed{\text{Def}}$ (Isomorphic representations)

Let G be a group. Two representations (φ, V) and (φ', V') of G are said to be isomorphic or similar, if there exists a vector space isomorphism $\psi: V \to V'$ s.t.

$$\varphi'(g) = \psi \circ \varphi(g) \circ \psi^{-1} \quad \text{for all } g \in G.$$

Ex.: $X = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ has eigenvalues $\exp(2\pi i k/d)$

for $k = 0, \ldots, d-1$. Hence, if $\omega = \exp(2\pi i/d)$ (primitive root of unity) then the unitary U diagonalizing X satisfies $\quad \varphi' = U \circ \varphi \circ U^\dagger$.

More examples of representations:

- Trivial representation: $\varphi(g) = 1_{\mathbb{F}}$ for all $g \in G$, where $\mathbb{F}$ is some field.

:) Regular representation of a finite group G:

Let $n = |G|$ and $V \cong \mathbb{C}^n$ with basis $\{|g\rangle\}_{g \in G}$.

Let $\varphi(g) : |h\rangle \mapsto |gh\rangle$ + linear extension.

(φ, V) is called the regular representation.

Conversely, let (ψ, W) be a rep. s.t. there exists $w \in W$

so that $\{\psi(g) w\}_{g \in G}$ is a basis of W. Then

ψ is isomorphic to the regular representation.

:) Permutation representation:

Let X be a finite set and G be a group acting on X.

Let $m = |X|$ and $V = \mathbb{C}^m$ with basis $\{|x\rangle\}_{x \in X}$.

Then $\varphi(g) : |x\rangle \mapsto |gx\rangle$ + linear extension defines

the permutation representation of G.

Regular representation of G:

permutation representation of G acting

on itself by left multiplication.

§1.2 Irreducible representations and decompositions

Def Let (φ, V) be a rep. of a group G. A subspace $W \leq V$ is called *invariant* or *stable* if $\varphi(g)|w\rangle \in W$ for all $|w\rangle \in W$ and $g \in G$.

The restriction $\varphi|_W$ of φ onto W is called a *subrepresentation*.

<u>Example:</u> Let G be a finite group with $n = |G|$ and $(\varphi, \mathbb{C}^n)$ be the regular representation.

Let $W = \operatorname{span}\left(\sum_{g \in G} |g\rangle\right)$. Then $(\varphi|_W, W)$ is a subrepresentation of $(\varphi, \mathbb{C}^n)$.

Note: $\{0\}$ and V are always invariant subspaces.

Def A representation (φ, V) is called *irreducible* if $\{0\}$ and V are the only invariant subspaces of V.

1-dimensional representations are always irreducible.

For example, W in the regular rep. above is irreducible.

Goal of representation theory: Find the irreducible representations of a group G!

Def (Direct sum of representations)

Let (φ_1, V_1) and (φ_2, V_2) be rep's of a group G. Then the direct sum $V_1 \oplus V_2$ affords the representation

$$[(\varphi_1 \oplus \varphi_2)(g)](v_1 \oplus v_2) := [\varphi_1(g)](v_1) \oplus [\varphi_2(g)](v_2)$$

of G called the direct sum representation.

A representation is called completely reducible, if it decomposes as a direct sum of irreducible representations.

Prop Let (φ, V) be a rep of a finite group G, where V is a vector space over a field whose characteristic does not divide the order of G. Then every G-invariant subspace W has a G-invariant complement W', i.e., $V = W \oplus W'$ (as vector spaces <u>and</u> as representations).

<u>Proof sketch:</u> Let P_W be the projection onto W and define

$$Q_W = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \circ P_W \circ \varphi(g)^{-1}.$$

Then you can check that $\operatorname{im} Q_W = W$, and $W' := \ker Q_W$ is the desired G-invariant complement. $\square$

Alternative proof of this proposition: Weyl's unitarity trick

Let (φ, V) be a representation over $\mathbb{C}$ and let

$\langle . | . \rangle : V \times V \to \mathbb{C}$ be an inner product on V.

Define the inner product $\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle \varphi(g) v | \varphi(g) w \rangle$

Then for every G-invariant subspace W the orthogonal complement $W^\perp$ (taken w.r.t. $\langle .,. \rangle_G$) is G-invariant as well, and $V = W \oplus W^\perp$ as representations.

Moreover, (φ, V) is a unitary representation w.r.t. $\langle .,. \rangle_G$, i.e., $\varphi(G) \leq \mathcal{U}(V)$, and $\varphi(g^{-1}) = \varphi(g)^{-1} = \varphi(g)^\dagger$.

For general unitary representations (φ, V) and an invariant subspace $W \leq V$, the orthogonal complement $W^\perp$ is again invariant.

$\boxed{\text{Prop}}$ (Maschke's theorem)

Every finite-dimensional representation of a finite group G over a field with characteristic not dividing $|G|$ is completely reducible.

Proof: Use induction over $\dim V$ and the preceding proposition. $\square$

G finite group, V finite-dim. representation over $\mathbb{C}$

Maschke $\Rightarrow$ $V = V_1 \oplus \ldots \oplus V_m$, with each V_i irreducible.

Is this unique?

$\boxed{\text{Prop}}$ (Schur's lemma)

Let (φ_1, V_1) and (φ_2, V_2) be irreducible representations of a group G, and let $f: V_1 \to V_2$ be a G-equivariant linear map: $f \circ \varphi_1(g) = \varphi_2(g) \circ f$ for all $g \in G$.

i) Either f is invertible (and hence $V_1 \cong V_2$) or $f = 0$.

ii) If $V_1 = V_2$ is finite-dim. over an algebraically closed field $\mathbb{F}$ (e.g. $\mathbb{F} = \mathbb{C}$), then $f = \lambda \mathbb{1}_{V_1}$ for some $\lambda \in \mathbb{F}$.

Proof: i) Let $f \neq 0$. Then $\ker f \neq V_1$ is a G-invariant subspace of V_1, so $\ker f = \{0\}$ by irreducibility of V_1. Likewise, $\operatorname{im} f \neq \{0\}$ is a G-invariant subspace of V_2, hence $\operatorname{im} f = V_2$, by irreducibility, and so f is invertible.

ii) Since $\mathbb{F}$ is algebraically closed, the linear map f has an eigenvalue, say $\lambda \in \mathbb{F}$. The map $f' = f - \lambda \mathbb{1}_{V_1}$

is G-equivariant, and by definition it is not invertible (since a non-zero eigenvector is in its kernel). It then follows from i) that $f' = 0$, i.e., $f = \lambda \mathbb{1}_{V_1}$. $\square$

$\boxed{\text{Cor}}$ Let G be an Abelian group, i.e., $gh = hg$ for all $g, h \in G$. Then any complex irreducible representation of G is 1-dimensional.

$\boxed{\text{Def}}$ (Isotypical component)

Let V be a finite-dimensional rep. of a finite group G over $\mathbb{C}$, and consider a decomposition $V = \bigoplus_k V_k$, where each V_k is the direct sum of n_k copies of an irreducible rep. W_k of G, i.e., $V_k = W_k \oplus .. \oplus W_k = W_k^{\oplus n_k} = W_k \otimes \mathbb{C}^{n_k}$, such that $W_k \not\cong W_j$ for $j \neq k$. Then V_k is called an isotypical component.

An application of Schur's lemma (see Teleman) shows:

$\boxed{\text{Prop}}$ The decomposition $V = \bigoplus_k V_k$ of a representation V into isotypical components V_k is unique, and so are the multiplicities n_k of W_k in V_k.