Lecture 37: Trace of matrices and operators

Last time: Jordan normal form

Def Trace of a matrix

Let $n \in \mathbb{N}$ and $A \in M_n(\mathbb{F})$ be a square matrix over some arbitrary field $\mathbb{F}$. The trace of A, denoted by $\operatorname{tr} A$, is the sum of its diagonal elements: $\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}$

$$A = \begin{pmatrix} A_{11} & & & A_{1n} \\ \vdots & A_{22} & & \\ & & \ddots & \\ A_{n1} & & & A_{nn} \end{pmatrix}$$

Ex.: $A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$, then $\operatorname{tr} A = 6$.

Prop 10.14 Let $A, B \in M_n(\mathbb{F})$ be two square matrices.

Then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Proof: If $A = (A_{ij})$, $B = (B_{ij})$, $1 \leq i,j \leq n$,

then $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$ and $(BA)_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj}$

$$\Rightarrow \operatorname{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik} = \sum_{k=1}^{n} (BA)_{kk} = \operatorname{tr}(BA)$$

$\square$

Def Two matrices $A, B \in M_n(\mathbb{F})$ are called similar, if there exists an invertible matrix $S \in M_n(\mathbb{F})$ s.t.

$$A = S \cdot B \cdot S^{-1}.$$

__Ex.:__ Let $T \in \mathcal{L}_{\mathbb{F}}(V)$ be an operator, and let B_1, B_2 be bases for V.

Then $\mathcal{M}(T)_{B_1,B_1}$ and $\mathcal{M}(T)_{B_2,B_2}$ are similar, because we have

$$\mathcal{M}(T)_{B_1,B_1} = S\ \mathcal{M}(T)_{B_2,B_2}\ S^{-1}$$

where $S = \mathcal{M}(I_V)_{B_2,B_1}$ is invertible with $S^{-1} = \mathcal{M}(I_V)_{B_1,B_2}$.

Prop If $A, B \in M_n(\mathbb{F})$ are similar, then $\operatorname{tr} A = \operatorname{tr} B$.

__Proof:__ Let $S \in M_n(\mathbb{F})$ be invertible s.t. $A = SBS^{-1}$.

Then $\operatorname{tr} A = \operatorname{tr}(S(BS^{-1})) = \operatorname{tr}((BS^{-1})S) = \operatorname{tr}(B(\underbrace{S^{-1}S}_{=I_n}))$

$= \operatorname{tr} B.$ $\square$

Def Trace of an operator

Let $T \in \mathcal{L}_{\mathbb{F}}(V)$. The trace of T, denoted $\operatorname{tr} T$, is defined as the trace of a matrix representation $\mathcal{M}(T)_{B,B}$ w.r.t. some basis B of V.

The above Prop. shows that this def. is independent of the chosen basis ($\mathcal{M}(T)_{B,B}$ and $\mathcal{M}(T)_{\tilde{B},\tilde{B}}$ are similar for bases B and $\tilde{B}$).

Prop Let $T \in \mathcal{L}_{\mathbb{F}}(V)$ be an operator over a VS V over $\mathbb{F}$, s.t. T has all $n = \dim V$ eigenvalues in $\mathbb{F}$ (e.g., $\mathbb{F} = \mathbb{C}$). Then the trace of T is equal to the sum of its eigenvalues (counted with multiplicity).

<u>Proof</u>: If T has all eigenvalues in $\mathbb{F}$, then adapting the proof of Prop 5.27 shows that there exists a basis $\mathcal{B}$ for V s.t.

$$\mathcal{M}(T)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of T.

Then, $\operatorname{tr} T = \operatorname{tr} \mathcal{M}(T)_{\mathcal{B},\mathcal{B}} = \lambda_1 + \dots + \lambda_n$. $\square$

Recall: If $\lambda_1, \dots, \lambda_m$ are the distinct eigenvalues of $T \in \mathcal{L}_{\mathbb{C}}(V)$ with multiplicities $d_i = \dim G(\lambda_i, T)$, $i = 1, \dots, m$, then the characteristic polynomial of T is

$$p_T = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}.$$

Prop 10.12 Let $T \in \mathcal{L}_{\mathbb{C}}(V)$ and $n = \dim V$. Then $-\operatorname{tr} T = -\sum_{i=1}^{m} d_i \lambda_i$ is the coefficient of z^{n-1} in the characteristic polynomial of T.

Proof: Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T with multiplicities $d_1, \dots, d_m$. Then

$$p_T = (z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$$
$$= z^n - \underbrace{(d_1\lambda_1 + \dots + d_m\lambda_m)}_{= \operatorname{tr} T} z^{n-1} + \dots \quad \Rightarrow \quad \square$$

Prop Properties of the trace

i) trace is linear: $\operatorname{tr}(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 \operatorname{tr} A_1 + \lambda_2 \operatorname{tr} A_2$

for $A_1, A_2 \in M_n(\mathbb{F})$, $\lambda_1, \lambda_2 \in \mathbb{F}$.

ii) $\operatorname{tr} A = \operatorname{tr} A^T$ for $A \in M_n(\mathbb{F})$

Let V be an inner product space:

iii) Let $U \leq V$ be a subspace of V and P_U be the corresponding orthogonal projection. Then $\operatorname{tr} P_U = \dim U$.

iv) Let $T \in \mathcal{L}(V)$ be self-adjoint. Then $\operatorname{tr} T \in \mathbb{R}$.

v) Let $T \in \mathcal{L}(V)$ be positive, then $\operatorname{tr} T \geq 0$.

vi) Let $T_1, T_2 \in \mathcal{L}(V)$ be positive operators, then $\operatorname{tr}(T_1 T_2) \geq 0$.

Proof: i), ii) are simple calculations

iii) Let $\dim U = m$, $\dim V = m+h$, and let $\{u_1, \dots, u_m\}$ and $\{w_1, \dots, w_h\}$ be orthonormal bases for U and $U^\perp$, resp.

Define $B = \{u_1, \dots, u_m, w_1, \dots, w_h\}$, which is a basis for V, then

$$\mathcal{M}(P_U)_B = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix} \Rightarrow \operatorname{tr} P_U = \operatorname{tr} \mathcal{M}(P_U)_B = \dim U.$$

$$\underbrace{\qquad}_{\dim U} \; \underbrace{\qquad}_{\dim U^\perp}$$

iv) Clear, since self-adjoint operators have real eigenvalues and hence tr is real as well.

v) Clear, since positive operators have non-neg. eigenvalues.

vi) Since T_1 is positive, it has a unique positive square root $\sqrt{T_1}$.

Claim: $\sqrt{T_1}\, T_2 \sqrt{T_1}$ is positive:

·) $\left(\sqrt{T_1}\, T_2 \sqrt{T_1}\right)^* = \left(\sqrt{T_1}\right)^* T_2^* \left(\sqrt{T_1}\right)^*$

$= \sqrt{T_1}\, T_2 \sqrt{T_1}$ ✓ s.-a.

·) Let $v \in V$ be arbitrary:

$$\langle \sqrt{T_1}\, T_2 \sqrt{T_1}(v), v \rangle = \langle T_2 \underbrace{\sqrt{T_1}(v)}_{=w}, \underbrace{\sqrt{T_1}(v)}_{=w} \rangle = \langle T_2(w), w \rangle \geq 0. \checkmark$$

But now, $\mathrm{tr}(T_1 T_2) = \mathrm{tr}\left(\sqrt{T_1}\sqrt{T_1}\, T_2\right)$

$$= \mathrm{tr}\left(\sqrt{T_1}\, T_2 \sqrt{T_1}\right) \overset{(v)}{\geq} 0 \qquad \square$$