Lecture 35: Characteristic and minimal polynomial

Last time: Generalized eigenspaces and multiplicities of eigenvalues

We let V be a finite-dim. VS over $\mathbb{C}$.

Def 8.34 Characteristic polynomial

Let $T \in \mathcal{L}(V)$ with distinct eigenvalues $\lambda_1, \dots, \lambda_m$ of multiplicities $d_i = \dim G(\lambda_i, T)$ for $i = 1, \dots, m$.

The polynomial $(z-\lambda_1)^{d_1} \cdot \ldots \cdot (z-\lambda_m)^{d_m}$ is called the characteristic polynomial of T.

Prop 8.36 Let $T \in \mathcal{L}(V)$.

i) The characteristic polynomial of T has degree equal to $\dim V$.

ii) The roots of the characteristic polynomial are the eigenvalues of T.

Proof: i) follows since $\sum_{i=1}^{m} d_i = \dim V$ (by Prop 8.26).

ii) follows by definition of the char. poly.. $\square$

Prop 8.37 Cayley-Hamilton theorem

Let q be the characteristic polynomial of an operator $T \in \mathcal{L}(V)$.

Then $q(T) = 0$.

Proof: Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of T with multiplicities $d_i = \dim G(\lambda_i, T)$, $i = 1, \dots, m$.

We know that $(T - \lambda_j I_V)|_{G(\lambda_j, T)}$ is nilpotent, so

$$(T - \lambda_j I_V)^{d_j}|_{G(\lambda_j, T)} = 0 \quad \text{by Prop 8.18.}$$

We have $V = G(\lambda_1, T) \oplus \dots \oplus G(\lambda_m, T)$, and so the claim

$$q(T) = 0 \iff q(T)(v) = 0 \ \forall v \in V$$

will follow if we can show that $q(T)|_{G(\lambda_j, T)} = 0 \ \forall j = 1, \dots, m$.

Recall that $q(z) = (z - \lambda_1)^{d_1} \dots (z - \lambda_m)^{d_m}$, so

$$q(T) = (T - \lambda_1 I_V)^{d_1} \dots (T - \lambda_m I_V)^{d_m} \quad \text{by Def. of } q.$$

Since all terms on the right-hand side commute, we can write for $v_j \in G(\lambda_j, T)$:

$$q(T)(v_j) = (T - \lambda_1 I_V)^{d_1} \dots (T - \lambda_{j-1} I_V)^{d_{j-1}} (T - \lambda_{j+1} I_V)^{d_{j+1}} \dots$$

$$\dots (T - \lambda_m I_V)^{d_m} \underbrace{(T - \lambda_j I_V)^{d_j} (v_j)}_{=0} = 0$$

$$\Rightarrow q(T)|_{G(\lambda_j, T)} = 0 \ \forall j = 1, \dots, m \Rightarrow q(T) = 0. \qquad \square$$

Minimal polynomial

Recall: a monic polynomial is one whose highest-degree coefficient is equal to 1.

$\boxed{\text{Prop 8.40}}$ Let $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial p of smallest degree such that $p(T) = 0$.

Proof: Let $n = \dim V$, then $\dim \mathcal{L}(V) = n^2$ $(= \dim M_n(\mathbb{C}))$
Hence, the n^2+1 operators $\{I_V, T, T^2, \dots, T^{n^2}\}$ are linearly dependent.
Let m be the smallest integer s.t. $\{I_V, T, T^2, \dots, T^m\}$ is linearly dependent (m could be n^2).

Then it follows that T^m is a linear combination of $\{T^j\}_{j=0}^{m-1}$:
there are $a_j \in \mathbb{C}$, $j = 0, \dots, m-1$, s.t.

$$a_0 I_V + a_1 T + a_2 T^2 + \dots + a_{m-1} T^{m-1} + T^m = 0$$

Let $p = a_0 + a_1 z + a_2 z^2 + \dots + a_{m-1} z^{m-1} + z^m \Rightarrow p(T) = 0$
Since m was the smallest integer with $\{1, T, \dots, T^m\}$ lin. dep.,
there is no other monic polynomial q of deg $< m$ with $q(T) = 0$.

Suppose q is another monic polynomial of deg $= m$ with $q(T) = 0$.

Then also $(p-q)(T) = \underbrace{p(T)}_{=0} - \underbrace{q(T)}_{=0} = 0$,

and since $\deg p = \deg q = m$ and both are monic, $\deg(p-q) < m$

$\Rightarrow p - q = 0$ by def. of m, and so $q = p$. $\square$

$\boxed{\text{Def. 8.43}}$ Minimal polynomial

Let $T \in \mathcal{L}(V)$. The minimal polynomial is the unique monic polynomial p of smallest degree s.t. $p(T) = 0$.

$\boxed{\text{Prop. 8.46}}$ Let $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbb{C})$ be a polynomial.

Then $q(T) = 0$ if and only if q is a (polynomial) multiple of the minimal polynomial of T.

Proof: ($\Leftarrow$) Let p denote minimal polynomial of T.

If $q = s \cdot p$ for some $s \in \mathcal{P}(\mathbb{C})$, then $q(T) = s(T) \cdot \underbrace{p(T)}_{=0} = 0$

($\Rightarrow$) Let $q(T) = 0$. By the division algorithm for polynomials (see textbook, Prop 4.8), there exist $s, r \in \mathcal{P}(\mathbb{C})$ s.t.

$$q = p \cdot s + r, \quad \text{and } \deg r < \deg p.$$

$q = ps + r$: $\quad 0 = q(T) = \underbrace{p(T)}_{=0} s(T) + r(T) = r(T)$

Hence, $r = 0$, since otherwise we could produce a monic polynomial $\tilde{r}$ with $\deg \tilde{r} < \deg p$ and $\tilde{r}(T) = 0$ ↯. $\Rightarrow q = p \cdot s$ $\square$

$\boxed{\text{Cor. 8.48}}$ The characteristic polynomial of an op. $T \in \mathcal{L}(V)$ is a (polynomial) multiple of the minimal polynomial.

Proof: Follows from Cayley-Hamilton thm (Prop. 8.37) and Prop. 8.46. $\square$

$\boxed{\text{Prop 8.49}}$ Let $T \in \mathcal{L}(V)$. The roots of the minimal polynomial are exactly the eigenvalues of T.

Proof: Let first $\lambda \in \mathbb{C}$ be a root of the minimal polynomial p of T.

Then $p(z) = (z - \lambda) q(z)$ where q is a monic polynomial of degree $< \deg p$. $\Rightarrow q(T) \neq 0$, i.e., there is $v \in V, v \neq 0$, s.t. $q(T)(v) = w \neq 0$.

Now, since $p(T) = 0$, we have
$$0 = (T - \lambda I_V)\, q(T)(v) = (T - \lambda I_V)(w),$$
i.e., λ is an eigenvalue of T with eigenvector w.

Let now $\lambda \in \mathbb{C}$ be an eigenvalue of T with eigenvector $u \neq 0$,

$T(u) = \lambda u$. $\Rightarrow$ $\underline{T^j(u) = \lambda^j u}$ $\forall j \in \mathbb{N}$.

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{m-1} z^{m-1} + z^m$ be the minimal polynomial of T. Then $0 = p(T)$, so

$$\underline{0} = p(T)(u)$$

$$= (a_0 I_V + a_1 T + a_2 T^2 + \dots + a_{m-1} T^{m-1} + T^m)(u)$$

$$= (a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_{m-1} \lambda^{m-1} + \lambda^m) u$$

$$\underline{= p(\lambda) u}$$

$\overset{u \neq 0}{\Rightarrow}$ $p(\lambda) = 0$, i.e., λ is a root of the minimal poly. p. $\square$