Lecture 26: Orthogonal complements

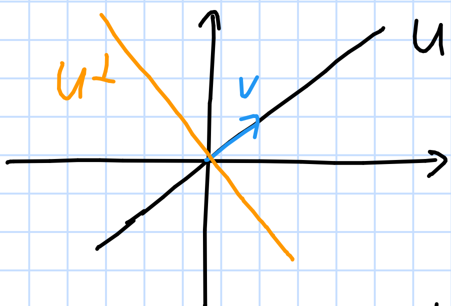
Last time: Orthonormal bases

V... inner product space with inner product $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$

Def 6.45 Orthogonal complement

Let $U \subseteq V$ be a subset of V. The orthogonal complement of U, denoted $U^\perp$, is the set of all vectors in V orthogonal to all vectors in U: $\quad U^\perp = \{ v \in V : \langle v, u \rangle = 0 \ \forall u \in U \}$

Ex.: $\mathbb{R}^2$ with the dot product (Euclidean / standard inner product)



$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\quad U = \langle v \rangle$

$\Rightarrow \quad U^\perp = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$

Let $w \in U^\perp$, $\ w = \lambda \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

Let $u \in U$, $\ u = \mu \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for some $\mu \in \mathbb{R}$.

$$\Rightarrow \langle w, u \rangle = \left\langle \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$$

$$= \lambda \mu \underbrace{\left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle}_{=0} = 0$$

Ex.: $\mathbb{R}^3$ with dot product

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 3x + 2y - z = 0 \right\} \Rightarrow U^\perp = \left\langle \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right\rangle$$

$\boxed{\text{Prop 6.46}}$ Properties of the orthogonal complement

i) For any subset $U \subseteq V$, the orthogonal complement $U^\perp$ is a subspace of V.

ii) $\{0\}^\perp = V$, $V^\perp = \{0\}$

iii) $U \cap U^\perp \subseteq \{0\}$ for any subset $U \subseteq V$.

(If $U \subseteq V$ is a subspace, then $U \cap U^\perp = \{0\}$)

iv) $U \subseteq W \subseteq V$: $W^\perp \subseteq U^\perp$.

<u>Proof</u>: i) Let $U \subseteq V$.

-) Since $\langle 0, u \rangle = 0 \ \forall u \in U \Rightarrow 0 \in U^\perp$

-) Let $v, w \in U^\perp$. Then for any $u \in U$,

$$\langle v + w, u \rangle = \underbrace{\langle v, u \rangle}_{=0} + \underbrace{\langle w, u \rangle}_{=0} = 0 \Rightarrow v + w \in U^\perp$$

-) Let $v \in U^\perp$, $\lambda \in \mathbb{C}$: Then for all $u \in U$,

$$\langle \lambda v, u \rangle = \lambda \underbrace{\langle v, u \rangle}_{=0} = \lambda \cdot 0 = 0 \Rightarrow \lambda v \in U^\perp$$

ii) $\langle v, 0 \rangle = 0 \ \forall v \in V \Rightarrow \{0\}^\perp = V$

Let $v \in V^\perp$, then by def. $\langle v, v \rangle = 0 \overset{\text{inner}}{\underset{\text{product}}{\Rightarrow}} v = 0 \Rightarrow V^\perp = \{0\}$.

iii) Let $v \in U \cap U^\perp$, then $\langle v, v \rangle = 0 \Rightarrow v = 0$

$$\Rightarrow U \cap U^\perp \subseteq \{0\} \quad \left(U \cap U^\perp = \emptyset \text{ if } 0 \notin U \right)$$

If $U \subseteq V$ is a subspace, then $0 \in U$, and $U \cap U^\perp = \{0\}$.

iv) Let $U \subseteq W \subseteq V$, and let $v \in W^\perp$. Then

$$\langle v, w \rangle = 0 \quad \forall w \in W,$$

in particular $\langle v, u \rangle = 0 \ \forall u \in U \subseteq W$, hence $v \in U^\perp$.

$\Rightarrow W^\perp \subseteq U^\perp$. $\square$

$\boxed{\text{Prop 6.47}}$ Direct sum of a subspace and its orthogonal comp.

Let V be finite-dim., and $U \subseteq V$ a subspace.

Then $V = U \oplus U^\perp$.

<u>Proof</u>: We first show that $V = U + U^\perp$.

Let $v \in V$ and $\{u_1, \dots, u_m\}$ be an ONB for U:

$$\langle u_i, u_j \rangle = \delta_{ij}.$$

Then $v = \underbrace{\langle v, u_1\rangle u_1 + \ldots + \langle v, u_m\rangle u_m}_{=:u} + \underbrace{v - \langle v, u_1\rangle u_1 - \ldots - \langle v, u_m\rangle u_m}_{=:w}$

$= u + w$

Clearly, $u \in \langle u_1, \ldots, u_m\rangle = U$.

To show: $w \in U^\perp$.

$1 \leq j \leq m$: $\quad \langle w, u_j\rangle = \langle v - \langle v, u_1\rangle u_1 - \ldots - \langle v, u_m\rangle u_m, u_j\rangle$

$$= \langle v, u_j\rangle - \langle v, u_1\rangle\langle u_1, u_j\rangle - \ldots - \langle v, u_m\rangle\langle u_m, u_j\rangle$$

$$= \langle v, u_j\rangle - \langle v, u_j\rangle \underbrace{\langle u_j, u_j\rangle}_{=1} = 0$$

$\Rightarrow \langle w, u\rangle = 0$ for all $u \in \langle u_1, \ldots, u_m\rangle = U$

$\Rightarrow w \in U^\perp$

$\Rightarrow V = U + U^\perp$.

By Prop 6.46 (iii), $U \cap U^\perp = \{0\}$, so $V = U \oplus U^\perp$. $\square$

$\boxed{\text{Cor 6.50}}$ If V is finite-dim., $U \leq V$, then

$$\dim U^\perp = \dim V - \dim U.$$

Prop 6.51 Let V be a finite-dim., $U \leq V$, then

$$U = (U^\perp)^\perp.$$

Proof: First show, that $U \subseteq (U^\perp)^\perp$:

Let $u \in U$, then $\langle u, v \rangle = 0 \ \forall v \in U^\perp \Rightarrow u \in (U^\perp)^\perp$ by def.

$\Rightarrow U \subseteq (U^\perp)^\perp$

In fact, $U \leq (U^\perp)^\perp$ is a subspace.

$V = U^\perp \oplus (U^\perp)^\perp$ by Prop 6.47,

so $\dim (U^\perp)^\perp = \dim V - \underbrace{\dim U^\perp}_{\dim V - \dim U} = \dim U$

by Cor. 6.50 $\Rightarrow U = (U^\perp)^\perp$. $\square$

Def 6.53 Orthogonal projection

Let V be a finite-dim. vector space, $U \leq V$ a subspace.

The orthogonal projection of V onto U, denoted P_U, is an operator in $\mathcal{L}(V)$ defined as follows: For $v \in V$, let $u \in U$ and $w \in U^\perp$ be the unique vectors s.t. $v = u + w$. Then $P_U(v) = u \in U$.