# Lecture 25: Orthonormal bases

Last time: Inequalities in inner product spaces

$\boxed{\text{Def}}$ Let $V$ be an inner product space with inner product $\langle .,. \rangle$ and $\|.\| = \sqrt{\langle .,. \rangle}$. Then $\{v_1, \dots, v_n\} \subseteq V$ is called orthonormal, if each $v_i$ has norm $1$ and the $v_i$ are pairwise orthogonal.

$\Leftrightarrow \quad \langle v_i, v_j \rangle = \delta_{ij} \quad 1 \leq i,j \leq n.$

Ex.: $V = \mathbb{C}^n, \quad \langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i} \quad , \quad x, y \in \mathbb{C}^n.$

·) The standard basis $\{e_1, \dots, e_n\}$ is orthonormal.

·) In $\mathbb{C}^3$, the vectors $v_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$

$\boxed{\text{Prop 6.25}}$ If $\{v_1, \dots, v_m\}$ is an orthonormal set of vectors, then $\|a_1 v_1 + \dots + a_m v_m\|^2 = |a_1|^2 + \dots + |a_m|^2$ for any $a_i \in \mathbb{C}$, $i = 1, \dots, m$.

Proof: $\| \sum_{i=1}^{m} a_i v_i \|^2 = \langle \sum_{i=1}^{m} a_i v_i , \sum_{j=1}^{m} a_j v_j \rangle$

$$= \sum_{i=1}^{m} a_i \sum_{j=1}^{m} \overline{a_j} \underbrace{\langle v_i, v_j \rangle}_{\delta_{ij}} = \sum_{i=1}^{m} \underbrace{a_i \overline{a_i}}_{|a_i|^2} = \sum_{i=1}^{m} |a_i|^2.$$

$\square$

**Prop 6.26** Let $\{v_1, \dots, v_m\}$ be orthonormal, then $\{v_1, \dots, v_m\}$ are also linearly independent.

<u>Proof</u>: Let $a_i \in \mathbb{C}$, $i = 1, \dots, m$ be such that

$$a_1 v_1 + \dots + a_m v_m = 0$$

Then $$0 = \|0\|^2 = \|a_1 v_1 + \dots + a_m v_m\|^2 \overset{\text{Prop 6.25}}{=} \sum_{i=1}^{m} |a_i|^2$$

Since $|a_i|^2 \geq 0 \ \forall i$, we have $|a_i|^2 = 0$, and thus $a_i = 0 \ \forall i = 1, \dots, m$.

$\Rightarrow \{v_1, \dots, v_m\}$ is lin. independent. $\square$

**Def 6.27** An orthonormal basis (ONB) of an inner product space $V$ is a basis consisting of orthonormal vectors.

<u>Ex.</u>: ·) The standard basis $\{e_1, \dots, e_n\}$ is an ONB for $\mathbb{C}^n$ with the usual Euclidean inner product.

·) In $\mathbb{C}^4$ (again with Euclid. inner product), $\{v_1, v_2, v_3, v_4\}$ is an ONB, where $v_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $v_2 = \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$, $v_3 = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$, $v_4 = \frac{1}{2}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$

If $\{v_1, \dots, v_n\}$ is an ONB of $V$, then for all $v \in V$,

$$v = \langle v, v_1 \rangle v_1 + \dots + \langle v, v_n \rangle v_n$$

and $$\|v\|^2 = |\langle v, v_1 \rangle|^2 + \dots + |\langle v, v_n \rangle|^2$$

(if $v = a_1 v_1 + \dots + a_n v_n$ for $a_i \in \mathbb{C}$, then $\langle v, v_j \rangle = a_1 \langle v_1, v_j \rangle + \dots + a_n \langle v_n, v_j \rangle$

$= a_j$ )

We can always turn an arbitrary basis of an inner product space into an ONB:

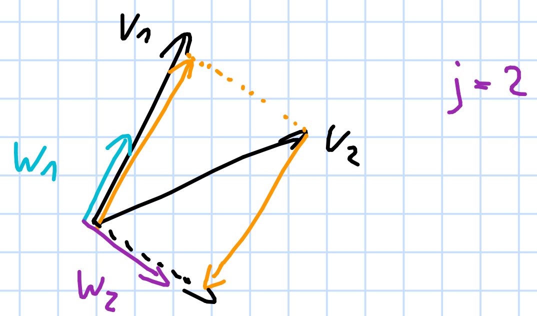
$\boxed{\text{Prop 6.31}}$ Gram-Schmidt procedure

Let $\{v_1, \dots, v_m\}$ be a set of linearly independent vectors.

Define $w_1 = \frac{1}{\|v_1\|} v_1$, and for $j = 2, \dots, m$

$$w_j = \frac{v_j - \langle v_j, w_1 \rangle w_1 - \dots - \langle v_j, w_{j-1} \rangle w_{j-1}}{\|v_j - \langle v_j, w_1 \rangle w_1 - \dots - \langle v_j, w_{j-1} \rangle w_{j-1}\|}$$

Then the $\{w_1, \dots, w_m\}$ are orthonormal, and $\langle v_1, \dots, v_m \rangle = \langle w_1, \dots, w_m \rangle$.



Proof: Induction on $j$:

$j=1$: $\quad w_1 = \frac{v_1}{\|v_1\|}$, and $\langle v_1 \rangle = \langle w_1 \rangle$

$j-1 \to j$: suppose that $w_1, \dots, w_{j-1}$ are orthonormal with

$$\langle w_1, \dots, w_{j-1} \rangle = \langle v_1, \dots, v_{j-1} \rangle$$

$v_j \notin \langle v_1, \dots, v_{j-1} \rangle = \langle w_1, \dots, w_{j-1} \rangle$, so

$$v_j - \langle v_j, w_1 \rangle w_1 - \dots - \langle v_j, w_{j-1} \rangle w_{j-1} \neq 0,$$

So we can set $w_j = \dfrac{v_j - \langle v_j, w_1 \rangle w_1 - \dots - \langle v_j, w_{j-1} \rangle w_{j-1}}{\| v_j - \langle v_j, w_1 \rangle w_1 - \dots - \langle v_j, w_{j-1} \rangle w_{j-1} \|}$

Then for $1 \leq l \leq j-1$,

$$\langle w_j, w_l \rangle = \frac{1}{\|\dots\|} \left( \langle v_j, w_l \rangle - \sum_{k=1}^{j-1} \langle v_j, w_k \rangle \overbrace{\langle w_k, w_l \rangle}^{\delta_{kl}} \right)$$

$$= \frac{1}{\|\dots\|} \left( \langle v_j, w_l \rangle - \langle v_j, w_l \rangle \right) = 0.$$

Moreover, $v_j \in \langle w_1, \dots, w_j \rangle$, and hence

$$\langle v_1, \dots, v_j \rangle \leq \langle w_1, \dots, w_j \rangle . \quad (*)$$

Since both $\{v_1, \dots, v_j\}$ and $\{w_1, \dots, w_j\}$ are lin. indep. ($v_k$'s by assumption, $w_k$'s by construction as orthonormal vec's), we have equality in $(*)$ $\square$

**Cor 6.34** Every finite-dim. inner product space has an ONB, and every list of orthonormal vectors can be extended to an ONB.

**Prop 6.37** Let $T \in \mathcal{L}_{\mathbb{F}}(V)$, where $V$ is an inner product space (and hence $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). If $T$ has an upper-triangular matrix representation w.r.t. some basis for $V$, then it also has an upper-triangular matrix rep. w.r.t. an ONB.

<u>Proof:</u> Recall (Prop 5.26): $T$ has an upper-triangular matrix rep. w.r.t. a basis $\{v_1, \dots, v_n\}$ iff $\langle v_1, \dots, v_j \rangle$ are invariant under $T$ for all $j = 1, \dots, n$.

Applying Gram-Schmidt proc. (Prop 6.31) to $\{v_1, \dots, v_n\}$ produces an ONB $\{w_1, \dots, w_n\}$, and $\langle w_1, \dots, w_j \rangle = \langle v_1, \dots, v_j \rangle$ is again invariant under $T \Rightarrow T$ is also upper-triangular w.r.t. ONB $\{w_1, \dots, w_n\}$. $\square$

**Cor 6.38** Schur's theorem

Every $T \in \mathcal{L}_{\mathbb{C}}(V)$ has an upper-triangular matrix rep. w.r.t. an ONB.