Lecture 24: Inequalities in inner product spaces

Last time: Inner product spaces

Def 6.11 Orthogonal vectors

Two vectors $v, w \in V$ in an inner product space $(V, \langle \cdot, \cdot \rangle)$ are called orthogonal, if $\langle u, v \rangle = 0$.

Note: The 0-vector in V is orthogonal to all other vectors ($\langle 0, v \rangle = \langle v, 0 \rangle = 0 \ \forall v \in V$), and it is the only vector orthogonal to itself ($\langle v, v \rangle = 0$ iff $v = 0$).

Prop 6.13 Pythagorean theorem

Let $(V, \langle \cdot, \cdot \rangle)$ be a real inner product space with norm $\|x\| = \sqrt{\langle x, x \rangle}$.
If $v, w \in V$ are orthogonal vectors, $\langle v, w \rangle = 0$, then

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2.$$

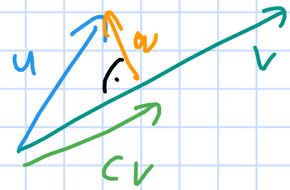
Proof: $$\|v+w\|^2 = \langle v+w, v+w \rangle = \langle v, v \rangle + \langle w, w \rangle + \overbrace{\langle v, w \rangle}^{=0} + \overbrace{\langle w, v \rangle}^{=0}$$
$$= \|v\|^2 + \|w\|^2 \qquad \square$$

Note: In a complex inner product space, we can have $\|v+w\|^2 = \|v\|^2 + \|w\|^2$ and $\langle v, w \rangle \neq 0$. Ex.: $u = \begin{pmatrix} 1 \\ i \end{pmatrix} \in \mathbb{C}^2$, $v = \begin{pmatrix} -2i \\ 3 \end{pmatrix} \in \mathbb{C}^2$

Prop 6.14 Let $u, v \in V$ be vectors in an inner product space $(V, \langle \cdot,\cdot \rangle)$, $v \neq 0$, and let $\|x\| = \sqrt{\langle x, x\rangle}$.

Setting $c = \frac{\langle u, v\rangle}{\|v\|^2}$ and $w = u - c\cdot v = u - \frac{\langle u, v\rangle}{\|v\|^2} v$.

Then $u = cv + w$, $\langle w, v\rangle = 0$.



Proof: $\langle w, v\rangle = \langle u - \frac{\langle u, v\rangle}{\|v\|^2} v, v\rangle$

$$= \langle u, v\rangle - \frac{\langle u, v\rangle}{\|v\|^2} \overbrace{\langle v, v\rangle}^{\|v\|^2} = 0$$

$\square$

Prop 6.15 Cauchy-Schwarz-inequality

Let $(V, \langle \cdot,\cdot \rangle)$ be an inner product space with $\|x\| = \sqrt{\langle x, x\rangle}$, and let $u, v \in V$: $|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$,

and we have equality if and only if u, v are scalar multiples of each other (i.e., linearly dependent).

Proof: If $v = 0$, then $0 = |\langle u, v\rangle| = \|u\| \cdot \|v\| = 0$.

Assume now $v \neq 0$, and set $w = u - \frac{\langle u, v\rangle}{\|v\|^2} v$ as in Prop 6.14.

Then $\langle v, w \rangle = 0$, and

$$\underline{\|u\|^2} = \left\| w + \frac{\langle u,v\rangle}{\|v\|^2} v \right\|^2 = \left\langle w + \frac{\langle u,v\rangle}{\|v\|^2} v , w + \frac{\langle u,v\rangle}{\|v\|^2} v \right\rangle$$

$$= \langle w,w\rangle + \cancel{\frac{\langle u,v\rangle}{\|v\|^2} \underbrace{\langle v,w\rangle}_{=0}} + \cancel{\frac{\overline{\langle u,v\rangle}}{\|v\|^2} \underbrace{\langle w,v\rangle}_{=0}}$$

$$+ \frac{\langle u,v\rangle}{\|v\|^2} \cdot \frac{\overline{\langle u,v\rangle}}{\|v\|^2} \langle v,v\rangle$$

$$\langle w,w\rangle \geq 0$$

$$= \|w\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^{\cancel{4}\,2}} \cancel{\|v\|^2}$$

$$\geq \frac{1}{\|v\|^2} |\langle u,v\rangle|^2 \xrightarrow[\text{by } \|v\|^2]{\text{mult.}} \text{claimed inequality.}$$

We have equality in the CS-inequality iff $\|w\|^2 = 0$ iff $w = 0$ iff $u = \frac{\langle u,v\rangle}{\|v\|^2} v = cv$ for some $c \in \mathbb{C}$. $\square$

Now, we can prove iii) in Prop from last lecture:

$\boxed{\text{Prop 6.18}}$ Let $(V, \langle \cdot,\cdot \rangle)$ be an inner product space, and set $\|x\| = \sqrt{\langle x,x\rangle}$ for $x \in V$. Then

$$\|x+y\| \leq \|x\| + \|y\| \quad \forall x,y \in V.$$

$\lambda \in \mathbb{C}$, $\lambda = a + bi$: $\mathrm{Re}(\lambda) = a \le \sqrt{a^2+b^2} = |\lambda|$

<u>Proof:</u> By Prop 6.15 (Cauchy-Schwarz),

$$|\langle x,y\rangle| \le \|x\|\cdot\|y\|$$

Now,
$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle y,y\rangle + \underbrace{\langle x,y\rangle + \overset{\overline{\langle x,y\rangle}}{\overset{\parallel}{\langle y,x\rangle}}}_{\substack{2\,\mathrm{Re}(\langle x,y\rangle) \\ \le 2|\langle x,y\rangle|}} \\
&\le \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle| \\
&\le \|x\|^2 + \|y\|^2 + 2\|x\|\cdot\|y\| \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

$\square$

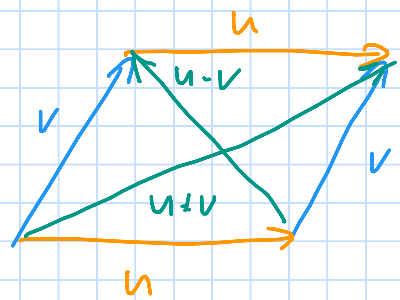
Note that we have equality in the triangle inequality iff $\langle x,y\rangle = \|x\|\cdot\|y\|$ iff one of u,v is a non-negative multiple of the other.

$\Rightarrow$ in an inner product space $(V, \langle\cdot,\cdot\rangle)$, the function $\|x\| = \sqrt{\langle x,x\rangle}$ always defines a norm.

Prop 6.22 Parallelogram equality

Let $(V, \langle .,. \rangle)$ be an inner product space, and let $\|x\| = \sqrt{\langle x,x \rangle}$ be the induced norm. Then for all $u, v \in V$ we have that

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right).$$



Proof:

$\|u+v\|^2 + \|u-v\|^2 =$

$$= \langle u+v, u+v \rangle + \langle u-v, u-v \rangle$$

$$= \|u\|^2 + \|v\|^2 + \cancel{\langle u,v \rangle} + \cancel{\langle v,u \rangle} + \|u\|^2 + \|v\|^2 - \cancel{\langle u,v \rangle} - \cancel{\langle v,u \rangle}$$

$$= 2\left(\|u\|^2 + \|v\|^2\right)$$ $\square$

Remark: ·) Polarization identity: If $\|x\| = \sqrt{\langle x,x \rangle}$ is a norm induced by an inner product, then

$$\langle x,y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|ix-y\|^2 - i\|ix+y\|^2\right).$$

·) Example of a norm <u>not</u> induced by an inner product:

1-norm on $\mathbb{C}^n$: $\|x\|_1 := \sum_{i=1}^{n} |x_i|$ (this is a norm!)

E.g. $\mathbb{C}^2$: $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$: $\|x\|_1^2 = 4 = \|y\|_1^2$, $\|x+y\|_1^2 = 4 = \|x-y\|_1^2$

$$8 = \|x+y\|_1^2 + \|x-y\|_1^2 \neq 2\left(\|x\|_1^2 + \|y\|_1^2\right) = 16.$$