Lecture 22: Eigenspaces and diagonalization

Last time: Existence of eigenvalues and upper-triangular matrices

Today: Is there an even simpler form of operators in terms of diagonal matrices?

$A \in M_n(\mathbb{F})$ diagonal: $A_{ij} = 0$ for $i \neq j$, $1 \leq i,j \leq n$.

$$A = \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & \ddots & \\ & & & * \end{pmatrix}$$

$\boxed{\text{Def 5.3}}$ Eigenspace

Let $T \in \mathcal{L}_{\mathbb{F}}(V)$ and $\lambda \in \mathbb{F}$. The eigenspace of T corresponding to λ is denoted $E(\lambda, T)$ and defined as

$$E(\lambda, T) = \ker(T - \lambda I_V)$$

This is a subspace of V (as the kernel of a linear map), and $E(\lambda, T) \neq \{0\}$ iff λ is an eigenvalue of T (because then $\exists\, v \in E(\lambda, T)$, $v \neq 0$, s.t. $(T - \lambda I_V)(v) = 0$ or $T(v) = \lambda v$).

Prop 5.38 Let V be finite-dim., $T \in \mathcal{L}_{\mathbb{F}}(V)$, and $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of T (i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$).

Then $E(\lambda_1, T) + \dots + E(\lambda_m, T)$ is a direct sum, and

$$\sum_{i=1}^{m} \dim E(\lambda_i, T) \leq \dim V.$$

Proof: Let $u_i \in E(\lambda_i, T)$ for $i = 1, \dots, m$ $\left(T(u_i) = \lambda_i u_i \right)$, s.t.

$$0 = u_1 + \dots + u_m.$$

Since the u_i are either zero or eigenvectors to distinct eigenvalues and hence linearly independent (Prop 5.10)

$$\rightarrow \quad u_i = 0 \text{ for all } i = 1, \dots, m$$

$\rightarrow \sum_{i=1}^{m} E(\lambda_i, T)$ is a direct sum, and

$$\dim \bigoplus_{i=1}^{m} E(\lambda_i, T) = \sum_{i=1}^{m} \dim E(\lambda_i, T) \leq \dim V. \qquad \square$$

Def 5.39 An operator $T \in \mathcal{L}_{\mathbb{F}}(V)$ is called diagonalizable, if there exists a basis B_V of V s.t. $A = \mathcal{M}(T)_{B_V, B_V}$ is diagonal, i.e., $A_{ij} = 0$ for $i \neq j$, $1 \leq i, j \leq \dim V$.

Prop 5.41 Conditions for diagonalizability

Let V be finite-dim. VS, $T \in \mathcal{L}_{\mathbb{F}}(V)$, and let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T.

TFAE:
i) T is diagonalizable.

ii) V has a basis consisting of eigenvectors of T.

iii) there are 1-dim subspaces $U_1, \dots, U_n$ of V s.t. each U_i is invariant under T ($T(U_i) \subseteq U_i$) and $V = U_1 \oplus \dots \oplus U_n$ ($\dim V = n$)

iv) $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$

v) $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$

Proof: i) $\Leftrightarrow$ ii): T is diag.ble iff $\exists\, \mathcal{B}_V = \{v_1, \dots, v_n\}$ s.t. $\mathcal{M}(T)_{\mathcal{B}_V, \mathcal{B}_V}$ is diagonal: $T(v_j) = \mu_j v_j$ for some $\mu_j \in \mathbb{F}$ ($\Rightarrow \mu_j = \lambda_j$)

ii) $\Rightarrow$ iii) Let $\{v_1, \dots, v_n\}$ be a basis for V with $T(v_j) = \lambda_j v_j$ for some $\lambda_j \in \mathbb{F}$. Define $U_j = \langle v_j \rangle$ for $j = 1, \dots, n$.

Then $\dim U_j = 1$ and $T(U_j) \subseteq U_j$, and $V = \bigoplus_{j=1}^{n} U_j$.

iii) => ii) Let $V = \bigoplus_{j=1}^{n} U_j$ for $U_j \leq V$, $\dim U_j = 1$,

and $T(U_j) \leq U_j$. Choose $v_i \in U_i$, $v_i \neq 0$, s.t. $U_i = \langle v_i \rangle$

Choose $w_i \in U_i$, $w_i \neq 0$, i.e., $w_i = a_i v_i$ for some $a_i \neq 0$.

Then $T(w_i) = T(a_i v_i) = a_i T(v_i) = c_i v_i$ for $c_i \in \mathbb{F}$.

$$\Rightarrow T(v_i) = \frac{c_i}{a_i} v_i \Rightarrow v_i \text{ is an eigenvec.}$$

Moreover, $\{v_1, \dots, v_n\}$ is a basis for V because $V = \bigoplus_{i=1}^{m} \langle v_i \rangle$.

$\Rightarrow$ i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii)

ii) => iv) If V has a basis of eigenvectors $\{v_1, \dots, v_n\}$,

then $v_i \in E(\lambda_j, T)$ for $i = 1, \dots, n$, $j = 1, \dots, m$, so

$$V = E(\lambda_1, T) + \dots + E(\lambda_m, T) \underset{\substack{\uparrow \\ \text{Prop 5.38}}}{=} E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$$

iv) => v) Clear by properties of a direct sum.

v) => ii) know: $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$

Choose bases for each $E(\lambda_i, T)$, $i = 1, \dots, m$.

Then we have a list $\{v_1, \dots, v_n\}$ of vectors in V, and $\dim V = n$.

$v_i \neq 0$ for $i = 1, \dots, m$, so all v_i's are eigenvectors of T.

To show that $\{v_1, \dots, v_n\}$ is a basis, we only need to show that they are linearly independent:

$$\text{Let } a_1 v_1 + \dots + a_n v_n = 0 \quad \text{for some } a_i \in \mathbb{F}. \qquad (*)$$

Let u_j $(j = 1, \dots, m)$ be the sum of those $a_h v_h$ that are in $E(\lambda_j, T)$. Then $(*)$ means that $u_1 + \dots + u_m = 0$ for $u_j \in E(\lambda_j, T)$.
But the $E(\lambda_j, T)$ form a direct sum (Prop 5.38), so each $u_j = 0$, and then all the a_h coefficients in the $a_h v_h$-terms for u_j are also zero (because the v_h's form bases for these spaces), and hence $a_h = 0$ for all $h = 1, \dots, n$.

$\Rightarrow \{v_1, \dots, v_n\}$ are lin. independent and hence a basis of eigenvectors $\Rightarrow$ ii) $\square$

Ex.: Not every operator is diagonalizable! (Not even over $\mathbb{C}$!)

$$T: \mathbb{C}^2 \to \mathbb{C}^2, \quad \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$$

Then $A = \mathcal{M}(T) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow 0$ is the only eigenvalue of T,

and $E(0, T) = \left\{ \begin{pmatrix} w \\ 0 \end{pmatrix} : w \in \mathbb{C} \right\}$ has dim. $1 \neq \dim \mathbb{C}^2 = 2$.

$\boxed{\text{Prop 5.44}}$ If $T \in \mathcal{L}_{\mathbb{F}}(V)$ with $\dim V = n$, and T has n distinct eigenvalues, then T is diagonalizable.

Proof: Since eigenvectors to distinct eigenvalues are lin. independent, V has a basis of eigenvectors of T, and hence T is diagonalizable by Prop 5.41. $\square$