Lecture 13: Linear maps as matrices

Last time: Fundamental theorem of linear algebra

$\boxed{\text{Def 3.30}}$ Matrices over a field

.) Let $\mathbb{F}$ be an arbitrary field and $m, n \in \mathbb{N}$

A $(m \times n)$-matrix $A = (A_{ij})$ is an array of elements in $\mathbb{F}$

with m rows and n columns:

$$A = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{pmatrix} \qquad A_{ij} \in \mathbb{F}, \quad 1 \leq i \leq m \text{ (row index)}$$
$$1 \leq j \leq n \text{ (col. index)}$$

The set of all $(m \times n)$-matrices over $\mathbb{F}$ is denoted by $M_{m,n}(\mathbb{F})$

or $\mathbb{F}^{m,n}$.

ii) We have a natural addition and scalar multiplication on $M_{m,n}(\mathbb{F})$

defined component-wise:

.) $A, B \in M_{m,n}(\mathbb{F})$: $\quad C = A + B \in M_{m,n}(\mathbb{F})$

where $C_{ij} = A_{ij} + B_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

.) $\lambda \in \mathbb{F}$, $A \in M_{m,n}(\mathbb{F})$: $\quad D = \lambda A \in M_{m,n}(\mathbb{F})$

where $D_{ij} = \lambda A_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Ex.: $A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\lambda = 2$

$$C = A + B = \begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}, \quad D = \lambda A = \begin{pmatrix} 4 & 0 \\ 6 & 2 \end{pmatrix}$$

$\boxed{\text{Prop 3.40}}$ Together with component-wise addition and scalar multiplication as defined in Def. 3.30 (ii), the set $M_{m,n}(\mathbb{F})$ of all $(m \times n)$-matrices over $\mathbb{F}$ is a vector space over $\mathbb{F}$ of dimension mn.

Proof: Vector space: simple exercise $\left(0_{M_{m,n}(\mathbb{F})} = \text{all-zeros matrix} \right)$

Straightforward to check that the following set of matrices is a basis for $M_{m,n}(\mathbb{F})$: $\{E_{ij}\}_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$

$$(E_{ij})_{kl} = \delta_{ik}\,\delta_{jl}$$

(that is, E_{ij} has a 1 in the i-th row and j-th col., and 0's elsewhere.)

Since there are $m \cdot n$ basis matrices, $\dim M_{m,n}(\mathbb{F}) = m \cdot n$ $\square$

Linear maps as matrices

Let $T: V \to W$ be a linear map between finite-dim. vector spaces V, W over a field $\mathbb{F}$.

Prop 3.5 says that, if $\{v_1, \dots, v_n\}$ is a basis for V, then T is uniquely defined by $\{T(v_1), \dots, T(v_n)\}$.

This allows us to assign a matrix to a linear map:

$\boxed{\text{Def. 3.32}}$ Matrix of a linear map

Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$ and fix bases $B_V = \{v_1, \dots, v_n\}$ of V and $B_W = \{w_1, \dots, w_m\}$ for W.

For all $j = 1, \dots, n$, let $A_{ij} \in \mathbb{F}$, $i = 1, \dots, m$, be such that

$$T(v_j) = A_{1j} w_1 + \dots + A_{mj} w_m .$$

Then the matrix associated to T with respect to the bases B_V and B_W, denoted $\mathcal{M}(T)_{B_V, B_W}$, is

$$\mathcal{M}(T) = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mn} \end{pmatrix}$$

$$\begin{array}{cccc} & T(v_1) & & T(v_m) \\ \mathcal{M}(T) = & \left(\begin{matrix} A_{11} \\ \vdots \\ A_{m1} \end{matrix}\right. & \begin{matrix} \dots \\ \\ \dots \end{matrix} & \left.\begin{matrix} A_{1n} \\ \vdots \\ A_{mn} \end{matrix}\right) \end{array}$$

The columns of $\mathcal{M}(T)$ are the vectors $T(v_j)$ (images of basis vectors in V) expanded in the basis B_W of W.

Often, we choose the standard basis in both V and W.

<u>Ex.:</u> Let $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_2 \end{pmatrix}$$

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis in $\mathbb{R}^2$ $(= S_V)$

$f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ — " — in $\mathbb{R}^3$ $(= S_W)$

$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = f_1 + f_2 \\ T(e_2) &= \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = f_1 - f_2 + f_3 \end{aligned}\right\} \quad \mathcal{M}(T)_{S_V, S_W} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 3 \end{pmatrix}$$

Choose now different bases: $B_V = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\}$, $B_W = \left\{ \overset{w_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \overset{w_2}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \overset{w_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$

$$\left.\begin{aligned} T(v_1) &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = w_1 \\ T(v_2) &= \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = 2w_2 + w_3 \end{aligned}\right\} \quad \mathcal{M}(T)_{B_V, B_W} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 3 \end{pmatrix}$$

$\boxed{\text{Prop}}$ Fix bases B_V for V and B_W for W.

i) $\mathcal{M}(T)_{B_V,B_W}$ uniquely defines a linear map $T: V \to W$ and vice versa:

$$S,T \in \mathcal{L}_{\mathbb{F}}(V,W): \quad S = T \iff \mathcal{M}(S)_{B_V,B_W} = \mathcal{M}(T)_{B_V,B_W}$$

ii) If $S,T \in \mathcal{L}_{\mathbb{F}}(V,W)$, then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$

$$\left(\mathcal{M}(\dots) = \mathcal{M}(\dots)_{B_V,B_W}\right)$$

iii) $\lambda \in \mathbb{F}$, $T \in \mathcal{L}_{\mathbb{F}}(V,W)$, then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$

Proof: i) follows immediately from Prop 2.29 (expansion of a vector in terms of a basis is unique) and Prop 3.5 (linear maps are uniquely defined by their images of vectors from a basis).

ii), iii): Recall: $\cdot$) $S,T \in \mathcal{L}_{\mathbb{F}}(V,W)$, then $S+T \in \mathcal{L}_{\mathbb{F}}(V,W)$,

where $(S+T)(v) = S(v) + T(v)$

$\cdot$) $\lambda \in \mathbb{F}$, $T \in \mathcal{L}_{\mathbb{F}}(V,W)$, then $\lambda T \in \mathcal{L}_{\mathbb{F}}(V,W)$,

where $(\lambda T)(v) = \lambda T(v)$.

Straightforward exercise to show i) and ii) $\square$