Lecture 12: Fundamental theorem of linear algebra

Last time: Kernel and image of a linear map

Recap: $T \in \mathcal{L}_{\mathbb{F}}(V, W)$

Kernel of T: $\ker T = \{v \in V : T(v) = 0\} \leq V$ (Prop 3.14)

Image of T: $\operatorname{im} T = \{w \in W : \exists v \in V : T(v) = w\} \leq W$ (Prop 3.19)

$\boxed{\text{Prop 3.22}}$ Fundamental theorem of linear algebra / linear maps

Let V, W be vector spaces over same field $\mathbb{F}$, $\dim V < \infty$,

and $T \in \mathcal{L}_{\mathbb{F}}(V, W)$. Then also $\operatorname{im} T$ is finite-dim., and

$$\dim V = \dim \ker T + \dim \operatorname{im} T$$

Proof: $\ker T \leq V$ is finite-dim., let $\{u_1, \dots, u_m\}$ be a basis for $\ker T$.

$\dim \ker T = m$. Extend this to a basis $\{u_1, \dots, u_m, v_1, \dots, v_n\}$ of V.

$\Rightarrow \dim V = m + n$.

Claim: $\{T(v_1), \dots, T(v_n)\}$ is a basis of $\operatorname{im} T$.

If claim is true, then $\dim V = m + n = \dim \ker T + \dim \operatorname{im} T$.

Proof of claim: Since $\{u_1, \dots, u_m, v_1, \dots, v_n\}$ is a basis of V,

we can write any $v \in V$ as $v = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$ for some $a_i, b_j \in \mathbb{F}$.

For all $v \in V$, $\quad v = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$ for some $a_i, b_j \in \mathbb{F}$.

apply T on both sides and use linearity:

$$T(v) = T\left(\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j\right)$$

$$= \cancel{\sum_{i=1}^{m} a_i \underbrace{T(u_i)}_{\substack{=0 \\ \text{since } u_i \in \ker T}}} + \sum_{j=1}^{n} b_j T(v_j) = \sum_{j=1}^{n} b_j T(v_j)$$

$\Rightarrow \langle T(v_1), \dots, T(v_n) \rangle = \operatorname{im} T$.

left to show: $\{T(v_1), \dots, T(v_n)\}$ are lin. indep.

Let $\sum_{j=1}^{n} c_j T(v_j) = 0$ for some $c_j \in \mathbb{F}$.

By linearity of T, $\quad T\Big(\underbrace{\sum_{j=1}^{n} c_j v_j}_{\in V}\Big) = 0$

That means: $\sum_{j=1}^{n} c_j v_j \in \ker T = \langle u_1, \dots, u_m \rangle$

But $\{u_1, \dots, u_m, v_1, \dots, v_n\}$ is a basis and hence lin. indep.

$\Rightarrow \sum_{j=1}^{n} c_j v_j \in \langle u_1, \dots, u_m \rangle$ only if $c_j = 0 \ \forall j$

$\Rightarrow \{T(v_1), \dots, T(v_n)\}$ is lin. indep. $\square$

$T \in \mathcal{L}_{\mathbb{F}}(V,W)$: $\quad \dim V = \dim \ker T + \dim \operatorname{im} T$

$\boxed{\text{Cor 3.23/24}}$ Let V, W be finite-dim. VS, and $T \in \mathcal{L}_{\mathbb{F}}(V,W)$.

i) If $\dim V > \dim W$, then T is not injective.

ii) If $\dim V < \dim W$, then T is not surjective.

<u>Proof:</u> i) $\operatorname{im} T \leq W \Rightarrow \dim \operatorname{im} T \leq \dim W$

$$\overset{\text{Prop 3.22}}{\Longrightarrow} \quad \dim \ker T = \dim V - \dim \operatorname{im} T$$

$$\geq \dim V - \dim W > 0$$

$\Rightarrow \ker T \neq \{0\}$ $(\dim \{0\} = 0)$ $\overset{\text{Prop 3.16}}{\Longrightarrow}$ T is not injective.

ii) $\dim \operatorname{im} T = \dim V - \underbrace{\dim \ker T}_{\geq 0} \leq \dim V < \dim W$

$\Rightarrow \operatorname{im} T \neq W$

$\Rightarrow T$ is not surjective $\quad \square$

Return to systems of (homogeneous) linear equations...

$$\begin{aligned} a_{11} x_1 + \ldots + a_{1n} x_n &= 0 \\ \vdots \qquad\qquad & \ \ \vdots \\ a_{m1} x_1 + \ldots + a_{mn} x_n &= 0 \end{aligned}$$

$m, n \in \mathbb{N}$

$a_{ij} \in \mathbb{F}$ for $1 \leq i \leq m$

$1 \leq j \leq n$

$$\begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \quad (*) \\ a_{m1}x_1 + \dots + a_{mn}x_n = 0 \end{array}$$

$m, n \in \mathbb{N}$

$a_{ij} \in \mathbb{F}$ for $1 \le i \le m$

$1 \le j \le n$

m eq's in n variables

homogeneous systems: $c = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{F}^n$ is always a solution.

Define a linear map $T: \mathbb{F}^n \to \mathbb{F}^m$ via the system $(*)$,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{F}^m, \quad y_i = \underbrace{\sum_{j=1}^{n} a_{ij}x_j}_{\text{LHS's of the rows of } (*)}$$

T is a linear map (HW4)

Above: c solution of $(*)$ $\iff$ $T(c) = 0$ $\;(= T(0))$

$\Rightarrow$ solution space of $(*)$ $=$ $\ker T$

$\boxed{\text{Prop 3.26}}$ Given a homogeneous system of linear eq's with n variables and m eq's. If $n > m$, then the space of solutions has dim. > 0 (dim ≥ 1). In particular, there are non-zero solutions to the system of lin. eq.

Proof: homogeneous SLE: $\sum_{j=1}^{n} a_{ij} x_j = 0$, $1 \leq i \leq m$

$a_{ij} \in \mathbb{F}$

$\Longleftrightarrow$ linear map $T : \mathbb{F}^n \to \mathbb{F}^m$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad y_i = \sum_{j=1}^{n} a_{ij} x_j$$

solution space $\left\{ c \in \mathbb{F}^n : \sum_{j=1}^{n} a_{ij} c_j = 0 \right\} = \ker T$.

Prop 3.22: $\dim \ker T = \dim \mathbb{F}^n - \underbrace{\dim \operatorname{im} T}_{\leq \dim \mathbb{F}^m = m}$

$$\geq n - m > 0 \qquad \square$$

Inhomogeneous systems of linear equations:

$$\left.\begin{array}{l} a_{11} x_1 + \ldots + a_{1n} x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1} x_1 + \ldots + a_{mn} x_n = b_m \end{array}\right. \quad (**) \quad \longleftrightarrow \quad T : \mathbb{F}^n \to \mathbb{F}^m$$

defined in terms of $(a_{ij})_{ij}$ as before

$$T(x) = b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{F}^m$$

$a_{ij} \in \mathbb{F}$

$b_i \in \mathbb{F}$

c is a solution for $(**)$

$\Longleftrightarrow \exists c \in \mathbb{F}^n : T(x) = b \Longleftrightarrow b \in \operatorname{im} T$.