Lecture 11: Kernel and image of a linear map

Last time: linear maps (definition, examples, properties)

$\boxed{\text{Def 3.12}}$ Kernel / null space of a linear map

Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$ be a linear map from V to W.

The kernel of T (null space of T), denoted $\ker T$ or null T, is the subset of V containing those vectors that are mapped to $0 \in W$ by T:

$$\ker T = \{ v \in V : T(v) = 0_W \}$$

Recall: $T(0) = 0$ for all linear maps $T \Rightarrow 0 \in \ker T$.

<u>Ex.</u>: ·) Let $T = 0$ be the zero map, $Tv = 0 \quad \forall v \in V$.

Then $\ker T = V$

·) Let $T = \mathrm{id} : V \to V$, $\ker T = \{0\} \subseteq V$

·) Let $T : \mathbb{F}^3 \to \mathbb{F}^1 = \mathbb{F}$, $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_1 + x_2 + x_3$

$\ker T$ is spanned by the vectors $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

$$\ker T = \langle v_1, v_2 \rangle$$

$\cdot$) Let $V = W = P_d(\mathbb{F})$ and consider $\partial : p \mapsto p'$ as a linear map. Then $\ker \partial = \{a \in \mathbb{F}\}$, the constant polynomials.

Prop 3.14 Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$, then $\ker T \leq V$.

Proof: Use Prop 1.34:

i) $0 \in \ker T$, since $T(0) = 0$ ✓

ii) let $u, v \in \ker T$, i.e., $T(u) = 0 = T(v)$

Then, $T(u+v) = T(u) + T(v) = 0 + 0 = 0 \Rightarrow u+v \in \ker T$. ✓

iii) $a \in \mathbb{F}$, $u \in \ker T$: $T(a \cdot u) = a\, T(u) = a \cdot 0 = 0 \Rightarrow au \in \ker T$. ✓

$\Rightarrow \ker T \leq V$. $\square$

Def 3.15 An arbitrary function $f : X \to Y$, where X and Y are sets, is called injective, if $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$, then $x_1 = x_2$.

($\Leftrightarrow$ every element in Y has at most one preimage)

Prop 3.16 A linear map $T: V \to W$ is injective if and only if $\ker T = \{0\}$.

Proof: ($\Rightarrow$) Let T be injective, and let $v \in \ker T$.

$$T(v) = 0 = T(0) \Rightarrow v = 0 \text{, since } T \text{ is injective.}$$
$$\Rightarrow \ker T = \{0\}.$$

($\Leftarrow$) Let $\ker T = \{0\}$, and assume that $T(v) = T(w)$, $v, w \in V$.

$$0 = T(v) - T(w) = T(v-w)$$
$$\Rightarrow v - w \in \ker T = \{0\} \Rightarrow v - w = 0 \Rightarrow v = w \Rightarrow T \text{ inj.} \quad \square$$

Def 3.17 image / range of a function

For an arbitrary function $f: X \to Y$, X, Y sets, the image of f (range of f), denoted $\operatorname{im} f$ or $\operatorname{ran} f$ ($\operatorname{range} f$), is defined as the set
$$\operatorname{im} f = \{ y \in Y : \exists x \in X \text{ w. } f(x) = y \}$$
$$= \{ f(x) : x \in X \}$$

Ex.: ·) For the zero map $T = 0$, $V \ni v \mapsto 0 \in W$,
$$\operatorname{im} T = \{0\} \leq W.$$

·) Identity map $id: V \to V$, $\operatorname{im}(id) = V \leq V$.

·) $T: \mathbb{F}^3 \to \mathbb{F}$, $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 + x_2 + x_3$

$\operatorname{im} T = \mathbb{F}$ (since $T\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a \in \mathbb{F}$)

·) $\partial : P_d(\mathbb{F}) \to P_d(\mathbb{F})$, $\partial : p \mapsto p'$

$\operatorname{im} \partial = P_{d-1}(\mathbb{F})$ (recall: $\ker \partial = \{a \in \mathbb{F}\} \cong \mathbb{F}$)

$$\left.\begin{aligned} \dim \ker \partial &= 1 \\ \dim \operatorname{im} \partial &= d \end{aligned}\right\} \text{ sum } = d+1 = \dim P_d(\mathbb{F})$$

$\boxed{\text{Prop 3.19}}$ Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$, then $\operatorname{im} T \leq W$.

<u>Proof</u>: Use Prop 1.34

i) $0_W \in \operatorname{im} T$ (since $T(0_V) = 0_W$) ✓

ii) $w_1, w_2 \in \operatorname{im} T$, then there are $v_1, v_2 \in V$ s.t. $T(v_i) = w_i$, $i=1,2$

$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2 \in \operatorname{im} T$ ✓

iii) $a \in \mathbb{F}$, $w \in \operatorname{im} T$, then there is $v \in V$ s.t. $T(v) = w$.

$T(av) = aT(v) = aw \in \operatorname{im} T$ ✓

$\Rightarrow \operatorname{im} T \leq W$. $\square$

$\boxed{\text{Def 3.20}}$ An arbitrary function $f: X \to Y$, X, Y sets

is called surjective, if $\operatorname{im} f = Y$

$\Leftrightarrow$ $\forall y \in Y \ \exists x \in X$ s.t. $f(x) = y$

$\Leftrightarrow$ every element in Y has at least one preimage.

Ex.: $\cdot)$ $id: V \to V$ has $\operatorname{im}(id) = V \Rightarrow id$ is surjective

$\cdot)$ $T: \mathbb{F}^3 \to \mathbb{F}$, $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 + x_2 + x_3$

$\operatorname{im} T = \mathbb{F} \Rightarrow T$ is surjective

Prop 3.19: $T: V \to W$ linear, then $\operatorname{im} T \leq W$

Prop 3.14: $T: V \to W$ linear, then $\ker T \leq V$