MATH 416 Abstract Linear Algebra

Homework Week 8 – October 14, 2021

Exercise 1 (2 points): Invariant subspaces
Let $S, T \in \mathcal{L}_{\mathbb{F}}(V)$. Prove the following statements:

(i) If $U_1, U_2 \leq V$ are invariant under T, then also $U_1 + U_2$ and $U_1 \cap U_2$ are invariant under T.

(ii) If $ST = TS$, then both $\ker S$ and $\operatorname{im} S$ are invariant under T.

Exercise 2 (4 points): Eigenvalues I
Find all eigenvalues and eigenvectors of the following operators over $\mathbb{F} = \mathbb{R}$ and $\mathbb{C}$:

(i) $S_1 \colon \mathbb{F}^2 \to \mathbb{F}^2, (x_1, x_2)^T \mapsto (-5x_2, 3x_1)^T$.

(ii) $S_2 \colon \mathbb{F}^3 \to \mathbb{F}^3, (x_1, x_2, x_3)^T \mapsto (2x_2, 0, 5x_3)^T$.

Exercise 3 (3 points): Eigenvalues II
Let $T \in \mathcal{L}_{\mathbb{F}}(V)$ be invertible.

(i) Show that T has no zero eigenvalues.

(ii) Let $\lambda \in \mathbb{F}$, $\lambda \neq 0$. Show that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(iii) Show that $v \in V$ is an eigenvector of T if and only if it is an eigenvector of T^{-1}.

Exercise 4 (6 points): Diagonalizable operators
Determine whether the following linear operators are diagonalizable, and if yes, give a basis for $\mathbb{C}^3$ consisting of eigenvectors (and verify that the vectors are eigenvectors).

$$\text{(i)} \quad R \colon \mathbb{C}^3 \to \mathbb{C}^3, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} 3z_1 + 3z_2 + 12z_2 \\ -2z_3 \\ -2z_3 \end{pmatrix}$$

$$\text{(ii)} \quad S \colon \mathbb{C}^3 \to \mathbb{C}^3, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} 3z_1 + 2z_2 \\ 3z_2 \\ z_3 \end{pmatrix}$$

$$\text{(iii)} \quad T \colon \mathbb{C}^3 \to \mathbb{C}^3, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} 2z_1 + z_2 + z_3 \\ z_2 \\ z_3 \end{pmatrix}$$