MATH 416 Abstract Linear Algebra

Homework Week 7 – October 7, 2021

Last update: October 13, 2021

Exercise 1 (7 points): Basis change matrices
Let $V = \mathbb{R}^3$, and consider the standard basis $\mathcal{S} = \{e_1, e_2, e_3\}$ and the bases $\mathcal{B} = \{v_1, v_2, v_3\}$ and $\mathcal{B}' = \{w_1, w_2, w_3\}$ with

$$v_1 = (1,1,1)^T \qquad v_2 = (1,-1,0)^T \qquad v_3 = (1,0,1)^T$$
$$w_1 = (1,0,1)^T \qquad w_2 = (1,-1,1)^T \qquad w_3 = (1,1,0)^T.$$

(i) (2 points) Compute $A = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$ and $B = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}}$ and verify that $B = A^{-1}$.

(ii) (2 points) Compute $C = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{S}}$ and $D = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}$ and verify that $D = C^{-1}$.

(iii) (2 points) Compute $E = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{B}'}$ and $F = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{B}}$ and verify that $F = E^{-1}$.

(iv) (1 point) What is the relationship between $\mathcal{M}(I_V)_{\mathcal{B},\mathcal{B}'}$, $\mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}$, and $\mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$?

Exercise 2 (3 points): Linear maps as matrices
Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2x - 3y \\ x + y + z \\ 3y - z \end{pmatrix},$$

and let $\mathcal{S}, \mathcal{B}, \mathcal{B}'$ be the bases from Exercise 1.

(i) (2 points) Determine $\mathcal{M}(T)_{\mathcal{S},\mathcal{S}}$ and $\mathcal{M}(T)_{\mathcal{B},\mathcal{B}'}$ using the definition of the matrix representation of a linear map.

(ii) (1 point) Verify that $\mathcal{M}(T)_{\mathcal{B},\mathcal{B}'} = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}\mathcal{M}(T)_{\mathcal{S},\mathcal{S}}\mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$.

Exercise 3 (2 points): Column span and surjectivity
Let V, W be finite-dimensional vector spaces over $\mathbb{F}$ with $n = \dim V$ and $m = \dim W$. Let $T \in \mathcal{L}_{\mathbb{F}}(V, W)$, and let $A = \mathcal{M}(T)_{\mathcal{B}_V,\mathcal{B}_W} \in M_{m,n}(\mathbb{F})$ for bases $\mathcal{B}_V$ and $\mathcal{B}_W$ for V, W, respectively. Prove that T is surjective if and only if the columns of A span $\mathbb{F}^m$.

Exercise 4 (3 points): Quotient spaces
Let $U, W \leq V$ be subspaces of a finite-dimensional vector space V such that $V = U \oplus W$. Show that $W \cong V/U$ by finding an explicit isomorphism.
Hint: Let $\{w_1, \ldots, w_m\}$ be a basis for W. Then show that $\{w_1 + U, \ldots, w_m + U\}$ is a basis for V/U and use this fact to construct an isomorphism.