MATH 416 Abstract Linear Algebra

Homework Week 6 – September 30, 2021

Exercise 1 (3 points): Matrix multiplication

(i) Find a matrix $P \in M_3(\mathbb{R})$, $P \notin \{0_3, I_3\}$, such that $P^2 = P$.

(ii) Find matrices $A, B \in M_2(\mathbb{R})$, $A, B \neq 0_2$, such that $AB = 0_2$.

(iii) Find a matrix $C \in M_3(\mathbb{R})$, $C \neq 0_3$, such that $C^3 = 0_3$.

Remark: Here, $0_n \in M_n(\mathbb{F})$ *for* $n \in \mathbb{N}$ *denotes the all-zeros matrix in* $M_n(\mathbb{F})$*.*

Exercise 2 (4 points): Invertibility of (2×2)-matrices

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{F}$ be an arbitrary (2×2)-matrix over a field $\mathbb{F}$. Use the inversion algorithm discussed in the lecture to:

(i) derive a necessary and sufficient condition (in terms of a, b, c, d) for A to be invertible;

(ii) give a formula for A^{-1} whenever it exists.

Exercise 3 (3 points): Elementary matrices

Determine elementary matrices $E_1, \ldots, E_k$ such that $A = E_k \ldots E_1$ for

$$A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 2 & -6 \\ 4 & -2 & 7 \end{pmatrix}.$$

Exercise 4 (5 points): Isomorphic vector spaces

Let V, W be finite-dimensional vector spaces over $\mathbb{F}$ with $\dim V = n$ and $\dim W = m$ for some $n, m \in \mathbb{N}$. Let $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ be an (ordered) basis for V, and $\mathcal{B}_W = \{w_1, \ldots, w_m\}$ an (ordered) basis for W. For $1 \leq i \leq m$ and $1 \leq j \leq n$, we define the linear maps $\phi_{ij} \colon V \to W$ via their action on the basis $\mathcal{B}_V$ by setting $\phi_{ij}(v_k) := \delta_{j,k} w_i$ for $1 \leq k \leq n$.

(i) Show that $\mathcal{L}_{\mathbb{F}}(V, W) = \langle \{\phi_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n} \rangle$, that is, every linear map $T \colon V \to W$ can be written as a linear combination of the maps ϕ_{ij}.

Hint: Expand $T(v_k)$ *in the basis* $\mathcal{B}_W$*. Can you derive a linear combination of T in terms of the* ϕ_{ij} *from this? Verify your result by checking that the resulting map agrees with T on the basis vectors* v_k*.*

(ii) Determine $\mathcal{M}(\phi_{ij})_{\mathcal{B}_V, \mathcal{B}_W}$.

(iii) Using the fact that $\dim \mathcal{L}_{\mathbb{F}}(V, W) = mn$, exercise (i) shows that $\{\phi_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n}$ is a basis for $\mathcal{L}_{\mathbb{F}}(V, W)$. Use this fact together with (ii) to construct an explicit isomorphism $\Phi \colon \mathcal{L}_{\mathbb{F}}(V, W) \to M_{m,n}(\mathbb{F})$.