MATH 416 Abstract Linear Algebra

Homework Week 15 – December 3, 2021

Exercise 1 (6 points):
Find the Jordan normal form and the characteristic and minimal polynomials of the following matrices over $\mathbb{C}$:

(i) $S = \begin{pmatrix} -2 & 5 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{pmatrix}$ (ii) $T = \begin{pmatrix} 1+i & 0 & 0 \\ 0 & 1-3i & 4i \\ 0 & -2i & 1+3i \end{pmatrix}$

Remark: You are only asked to give the matrix of the Jordan normal form, not the corresponding basis.

Exercise 2 (9 points):
For the matrices (in Jordan normal form) below, list the following data in each case: eigenvalues, their algebraic and geometric multiplicities, the characteristic polynomial, the minimal polynomial, the determinant, the trace

$$A = \begin{pmatrix} 4 & 1 & . & . & . & . & . & . \\ . & 4 & 1 & . & . & . & . & . \\ . & . & 4 & . & . & . & . & . \\ . & . & . & 4 & 1 & . & . & . \\ . & . & . & . & 4 & . & . & . \\ . & . & . & . & . & -3 & 1 & . \\ . & . & . & . & . & . & -3 & . \\ . & . & . & . & . & . & . & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & . & . \\ . & 0 & . & . \\ . & . & 0 & 1 \\ . & . & . & 0 \end{pmatrix} \quad C = \begin{pmatrix} i & . & . & . & . \\ . & i & . & . & . \\ . & . & -i & . & . \\ . & . & . & -i & . \\ . & . & . & . & -i \end{pmatrix}$$

Remark: The zero elements in the matrices are denoted by a dot.

Exercise 3 (4 points): Minimal polynomial

(i) Let $T \in \mathcal{L}_{\mathbb{C}}(V)$ be an operator. Show that T is invertible if and only if the constant term of the minimal polynomial of T is non-zero.

(ii) Let V be an inner product space and $T \in \mathcal{L}(V)$. Let

$$q_T = a_0 + a_1 z + a_2 z^2 + \cdots + a_{k-1} z^{k-1} + z^k$$

be the minimal polynomial of T. Show that the minimal polynomial of the adjoint operator T^* is given by

$$q_{T^*} = \overline{a_0} + \overline{a_1}z + \overline{a_2}z^2 + \cdots + \overline{a_{k-1}}z^{k-1} + z^k.$$

Exercise 4 (3 points): Trace

(i) Show that the space of traceless matrices, $\{A \in M_n(\mathbb{F})\colon \operatorname{tr}(A) = 0\}$, is a subspace of $M_n(\mathbb{F})$. What is its dimension?

(ii) Let V be an inner product space and $A \in \mathcal{L}(V)$ be such that $\operatorname{tr}(AB) = 0$ for all $B \in \mathcal{L}(V)$. Show that $A = 0$.

Exercise 5 (3 points): Functions on matrices

(i) By the uniqueness of the determinant, we know that $f\colon M_2(\mathbb{F}) \to \mathbb{F}$, $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad$ is not a determinant function. Which property of the determinant fails to hold for f?

(ii) The same for $g\colon M_2(\mathbb{F}) \to \mathbb{F}$, $g\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a - b$.

(iii) The same for $h\colon M_2(\mathbb{F}) \to \mathbb{F}$, $h\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3(ad - bc)$.