MATH 416 Abstract Linear Algebra

Homework Week 11 – November 4, 2021

Exercise 1 (2 points): Adjoint maps

(i) Let $T \in \mathcal{L}(V)$ and U be a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

(ii) Let $T \in \mathcal{L}(V, W)$. Show that T is injective if and only if T^* is surjective.

Exercise 2 (4 points): Self-adjoint maps and Pauli matrices
Consider the Pauli matrices $I, X, Y, Z \in M_2(\mathbb{C})$, defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that $\{I, X, Y, Z\}$ is a basis for the real vector space $\mathcal{S}_2(\mathbb{C}) = \{A \in M_2(\mathbb{C}) \colon A^* = A\}$ of self-adjoint complex (2×2)-matrices.

Exercise 3 (4 points): Normal operators
Consider the basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ defined as $v_1 = (1, 0, -2)^T$, $v_2 = (0, 3, 0)^T$, $v_3 = (2, 0, 1)^T$. Let $\mathcal{S}$ be the standard basis of $\mathbb{R}^3$.

(i) Show that $M = \mathcal{M}(I)_{\mathcal{B},\mathcal{S}}$ is a normal matrix, i.e., $M^*M = MM^*$.

(ii) Let now $\mathcal{C} = \{c_1v_1, c_2v_2, c_3v_3\}$ for some $c_i \neq 0$. Find c_1, c_2, c_3 such that $N = \mathcal{M}(I)_{\mathcal{C},\mathcal{S}}$ satisfies

$$NN^* = N^*N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 4 (5 points): Spectral theorem
Consider the self-adjoint matrix

$$A = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix}.$$

(i) Find the eigenvalues of A and an orthonormal basis $\mathcal{B}$ for $\mathbb{C}^2$ consisting of eigenvectors.

(ii) Let $U = \mathcal{M}(I)_{\mathcal{B},\mathcal{S}}$, and compute U^*AU. What do you find?