Quantum Capacity Bounds and Semidefinite Programming

Members: Mayank Bhatia, Mason Camp, Yuxuan Chen, Paul Ge, Evan Papoutsis, John Solak, Jonathan Wang, Tianfan Xu, Boqin Yuan
Project Leaders: Yidong Chen, Joe Groszkiewicz, Peixue Wu
Faculty Mentors: Roy Araiza, Felix Leditzky

Abstract
We evaluate the upper bound on the quantum capacity of two quantum channels using semi-definite programming. We present data investigating the bounds and simplify our approach to linear programming.

1. Background

1.1 Entanglement

Entanglement is a physical phenomenon that occurs when a group of particles interact in a way such that a single particle shares a non-trivial, genuinely quantum type of correlation with the other particles.

1.2 Noise & Quantum Channels

Noise in a quantum system can be understood as an unwanted interaction with the environment that hinders quantum communication and computation. A **quantum channel** is a mathematical model for noise in quantum systems.

1.3 Generating entanglement using quantum channels

When a quantum channel is not too noisy, it may be used to send quantum information, which is equivalent to generating entanglement [3].

2. Problem Statement

2.1 Quantum Capacity

Capacities of a noisy communication channel are a measure of the channel's usefulness for information transmission.

The quantum capacity of a quantum channel $\mathcal{N}$, denoted $\mathcal{Q}(\mathcal{N})$, characterizes the highest rate at which entanglement can be generated using $\mathcal{N}$ [3].

Unfortunately, finding $\mathcal{Q}(\mathcal{N})$ is an unbounded optimization problem that we do not know how to solve in general. We can, however, analyze bounds of $\mathcal{Q}(\mathcal{N})$.

2.2 Covariant Quantum Channels

We will analyze the quantum capacity of two symmetric quantum channels: the Depolarizing and Werner-Holevo channels.

A Depolarizing channel acts by either preserveing a quantum state or replacing it with the maximally mixed state:

$$\mathcal{D}_q(\rho) = (1-q)\rho + q\,\mathrm{tr}(\rho)\tfrac{1}{d}\mathbb{1} \quad \text{for } q \in [0, d^2/(d^2-1)].$$

A Werner-Holevo channel acts by either transposing the state or replacing it with the maximally-mixed state:

$$\mathcal{W}_p(\rho) = (1-p)\rho^T + p\,\mathrm{tr}(\rho)\tfrac{1}{d}\mathbb{1} \quad \text{for } p \in [d/(d+1), d/(d-1)].$$

3. Upper Bound on the Quantum Capacity

3.1 Choi Operator

We can determine an upper bound on the quantum capacity in terms of the Choi operator of a channel, which is obtained by letting the channel act on one half of a so-called maximally entangled state [2,4].

$$\tau(\mathcal{N}) = \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{N}(|i\rangle\langle j|) \quad \text{for quantum channel } \mathcal{N}.$$

3.2 Upper Bound Using Semi-definite Programming

Wang et al. derived an upper bound $\log\Gamma(\mathcal{N})$ on a quantum channel $\mathcal{N}$ that can be computed by solving a specific family of convex optimization problems called semidefinite programs (SDP) [5].

Proposition 1. *Let $\mathcal{N}: A \to B$ be a quantum channel with Choi operator $\tau(\mathcal{N})$. Then* $\mathcal{Q}(\mathcal{N}) \le \log\Gamma(\mathcal{N})$, *where $\Gamma(\mathcal{N})$ is the solution of the following semidefinite program:*

minimize $\mu \in \mathbb{R}$ *subject to*
$$V_{AB}, Y_{AB} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_B),$$
$$(V_{AB} - Y_{AB})^{T_B} \ge \tau(\mathcal{N}),$$
$$V_{AB} + Y_{AB} \le \mu \mathbb{1}_A,$$

with $\mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_B)$ representing the set of positive semidefinite operators acting on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.

3.3 Result: Numerically solving the SDP for $\Gamma(\mathcal{N})$

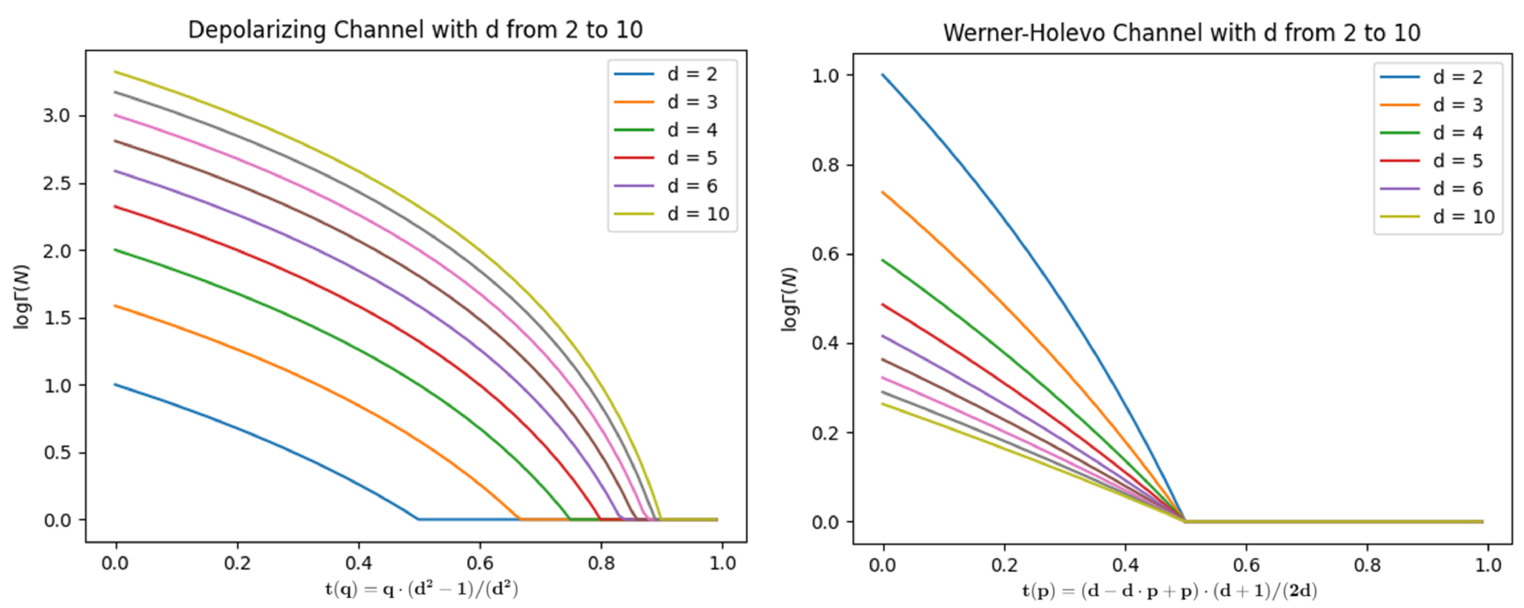

Figure 1: *Upper bound of capacity for Depolarizing and Holevo-Werner channels, where d is the dimension of the quantum system.*

4. Simplifying the Problem

4.1 Symmetries

The Depolarizing and Werner-Holevo channels both have nice symmetries that make their Choi representation invariant. $\forall U$ in the unitary group acting on $\mathbb{C}^d$,

$$\tau(\mathcal{D}_q) = (U \otimes \overline{U})\,\tau(\mathcal{D}_q)\,(U \otimes \overline{U})^\dagger$$
$$\tau(\mathcal{W}_p) = (U \otimes U)\,\tau(\mathcal{W}_p)\,(U \otimes U)^\dagger$$

By using this symmetry of unitaries, we can reduce the problem to a linear program, allowing us to analyze bounds on the quantum capacities of our channels in higher dimensions.

4.2 Upper Bound Using Linear Programming

If $(V_{AB},\ Y_{AB},\ \mu)$ is an optimal solution of the SDP, then so is

$$((U \otimes U)V_{AB}(U \otimes U)^\dagger,\ (U \otimes U)Y_{AB}(U \otimes U)^\dagger,\ \mu).$$

One can imagine taking an "average" over the unitary group, resulting in

$$\left(\int_{U(d)} (U \otimes U)V_{AB}(U \otimes U)^\dagger d\mu(U),\ \int_{U(d)} (U \otimes U)Y_{AB}(U \otimes U)^\dagger d\mu(U),\ \mu\right).$$

This remains optimal because it is a convex combination of optimal solutions.

Proposition 2. *For any Hermitian operator $T_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$, there exist $t_1, t_2 \in \mathbb{R}$, such that:*

$$\int_{U(d)} (U \otimes U)T_{AB}(U \otimes U)^\dagger d\mu(U) = t_1(1_d \otimes 1_d) + t_2 \cdot \mathbb{F}_{AB}$$

Proposition 3. $\log\Gamma(\mathcal{N})$ *for Depolarizing channels from proposition 1 can be obtained by the following linear program:*

minimize $\mu \in \mathbb{R}$ *subject to*
$$r_1, r_2, r_3, r_4 \in \mathbb{R},$$
$$r_1 + r_2 \ge 0, r_3 + r_4 \ge 0,$$
$$r_1 - r_2 \ge 0, r_3 - r_4 \ge 0,$$
$$r_1 - r_3 - \tfrac{q}{d} + d(r_2 - r_4 - 1 + q) \ge 0,$$
$$r_1 - r_3 - \tfrac{q}{d} \ge 0,$$
$$d(r_1 + r_3) + r_2 + r_4 \le \mu.$$

We can obtain a linear program for Werner-Holevo channels similarly.

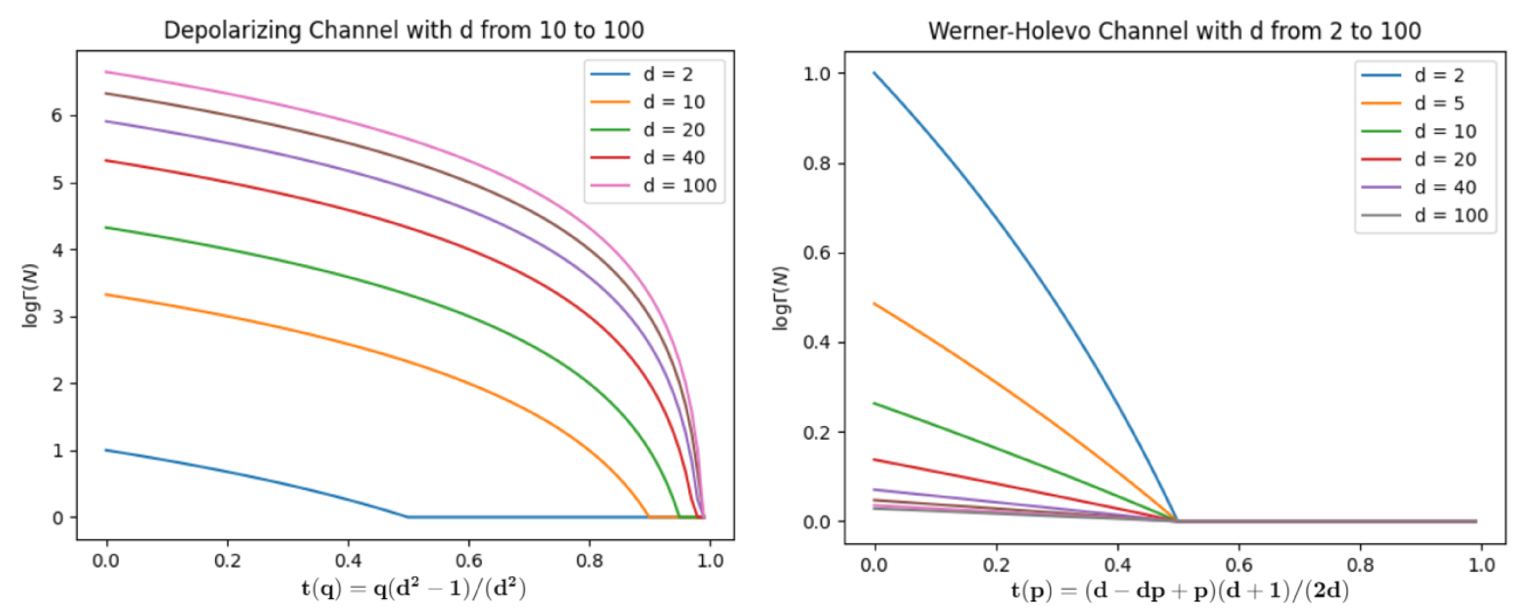

Figure 2: *Upper bound of capacity for Depolarizing and Holevo-Werner channels, generated using linear programming.*

5. Future Goals

Using their symmetries, we've finished reducing the quantum capacity bound SDP for the Depolarizing and Werner-Holevo channels to a linear program. This allows us to evaluate the bounds in higher dimensions, where we hope to explore the asymptotics of $\Gamma(\mathcal{N})$ in the limit of large d and prove why these channels demonstrate such behaviors.

References

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