Math 595 Quantum channels

Exercise sheet 3 – March 23, 2023

Unless stated otherwise, $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \dots$ denote finite-dimensional Hilbert spaces.

1. Show that any complementary channel of a generalized dephasing channel is entanglement-breaking. Use this to show that generalized dephasing channels are degradable.

2. Let $\vartheta\colon X \mapsto X^T$ denote the transposition map, and let $\mathcal{N}$ be an arbitrary CP map. Show that $\theta \circ \mathcal{N} \circ \theta$ is CP.

3. Let $\mathcal{A}_\gamma$ with $\gamma \in [0,1]$ be the amplitude damping channel with Kraus operators $K_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $K_1 = \sqrt{\gamma}|0\rangle\langle 1|$.

   (a) Let $\gamma \in [0,1/2]$ and set $\delta = \frac{1-2\gamma}{1-\gamma}$. Show that $\mathcal{A}_{1-\gamma} = \mathcal{A}_\delta \circ \mathcal{A}_\gamma$.

   (b) Show that $\mathcal{A}_\gamma^c = \mathcal{A}_{1-\gamma}$ for all $\gamma \in [0,1]$ is a possible choice of complementary channel.

   (c) Use these results to argue that $\mathcal{A}_\gamma$ is degradable for $\gamma \in [0,1/2]$ and antidegradable for $\gamma \in [1/2,1]$.

4. Show that the amplitude damping channel $\mathcal{A}_\gamma$ is covariant with respect to the group $G = \{\mathbb{1}, Z\}$ (and identical representations on the input and output space).

5. The von Neumann entropy S is defined for quantum states $\rho \in \mathcal{B}(\mathcal{H}), \rho \geq 0, \operatorname{tr}\rho = 1$ as $S(\rho) = -\operatorname{tr}\rho\log\rho$.[1] One of its properties is *subadditivity*:[2]

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \tag{1}$$

   (a) Let $\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x$ be a classical-quantum state. Prove that

$$S(\rho_{XA}) = H(\{p_x\}) + \sum_x p_x S(\rho_A^x), \tag{2}$$

   where $H(\{p_x\}) = -\sum_x p_x \log p_x$ is the Shannon entropy.

   (b) Use (1) and (2) to prove concavity of the von Neumann entropy:

$$S\Big(\sum_x p_x \rho_x\Big) \geq \sum_x p_x S(\rho_x).$$

[1] Let $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ be a spectral decomposition of the state ρ. Then the operator $\log\rho$ is defined via spectral calculus:

$$\log\rho = \sum_{i\colon \lambda_i > 0} \log(\lambda_i)|\psi_i\rangle\langle\psi_i|.$$

[2] We will prove a stronger form of this property called *strong subadditivity* in the second half of the course.

Hint: Assemble the state ensemble $\{p_x, \rho_x\}$ in a classical-quantum state ρ_{XA} and evaluate both sides of (1) for this state.

6. Let $|\Phi^+\rangle_{AA'} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_A \otimes |i\rangle_{A'}$ with $d = \dim \mathcal{H}_A = \dim \mathcal{H}_{A'}$ be a maximally entangled state, and let $\mathcal{N} \colon A \to A$ be a quantum channel. Then the *entanglement fidelity* $F(\mathcal{N})$ is defined as

$$F(\mathcal{N}) = \langle \Phi^+ | (\mathrm{id}_A \otimes \mathcal{N})(\Phi^+) | \Phi^+ \rangle.$$

There is a related quantity called the *average fidelity*: let $\int_{\mathcal{U}(d)} dU$ denote the Haar integral with respect to the Haar measure on the unitary group $\mathcal{U}(d)$ on $\mathcal{H}_A$, and for a channel $\mathcal{N} \colon A \to B$ define the average fidelity $f(\mathcal{N})$ as

$$f(\mathcal{N}) = \int_{\mathcal{U}(d)} dU \langle \phi | U^\dagger \mathcal{N}(U \phi U^\dagger) U | \phi \rangle,$$

where $|\phi\rangle \in \mathcal{H}_A$ is a fixed but arbitrary pure state.

(a) Show that both $f(\mathcal{N})$ and $F(\mathcal{N})$ are invariant under channel twirling, i.e., $f(\mathcal{N}) = f(\mathcal{N}_{\mathcal{U}(d)})$ and $F(\mathcal{N}) = F(\mathcal{N}_{\mathcal{U}(d)})$ where $\mathcal{N}_{\mathcal{U}(d)}(X) = \int_{\mathcal{U}(d)} dU\, U^\dagger \mathcal{N}(UXU^\dagger) U$.

(b) Verify that a depolarizing channel $\mathcal{D}_q \colon \rho \mapsto (1-q)\rho + q \operatorname{tr}(\rho) \frac{1}{d} \mathbb{1}_A$ satisfies the following identity for entanglement fidelity and average fidelity:

$$f(\mathcal{D}_q) = \frac{F(\mathcal{D}_q) d + 1}{d+1}. \tag{3}$$

(c) Use Exercises 6a and 6b to show that (3) holds for *arbitrary* channels $\mathcal{N}$:

$$f(\mathcal{N}) = \frac{F(\mathcal{N}) d + 1}{d+1}.$$

Hint: What do we know about the channel $\mathcal{N}_{\mathcal{U}(d)}$ in 6a?

7. Let G be a compact group and $g \mapsto U_g$ be an irreducible unitary representation on $\mathcal{H}_A$. Show that, for any $X_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$,

$$\frac{1}{|G|} \sum_{g \in G} (U_g \otimes \mathbb{1}_B) X_{AB} (U_g \otimes \mathbb{1}_B)^\dagger = \frac{1}{|A|} \mathbb{1}_A \otimes X_B,$$

where $X_B = \operatorname{tr}_A X_{AB}$.