# Recap

- Accessible information: measure classical information $X \sim p_X$ encoded in a quantum state ensemble $\{p_x, \rho_B^x\}$ using a POVM $E = \{E_B^y\} \longrightarrow$ RV $Y$. $\quad I_{acc}(\{p_x, \rho_B^x\}) = \max_E I(X;Y)$

- Holevo bound: $I_{acc}(\{p_x, \rho_B^x\}) \leq I(X;B) \leq \log|B|$

  $\Rightarrow$ cannot encode more than $n$ classical bits in $n$ qubits!

- Conditional mutual information:

$$I(A;B|C) = S(A|C) + S(B|C) - S(AB|C)$$

$$= S(AC) + S(BC) - S(C) - S(ABC)$$

- $I(A;B|C) \geq 0$ by strong subadditivity

- When is $I(A;C|B) = 0$?

- Classical Markov chain $X \to Y \to Z$: $X$ and $Z$ are independent conditioned on $Y$.

$\Leftrightarrow$ $p_{XZ|Y} = p_{X|Y}\, p_{Z|Y}$ $\Leftrightarrow$ $I(X;Z|Y) = 0$

$\Leftrightarrow$ $\exists$ channel $W: Y \to Z$ s.t. $p_{XYZ} = W_{Z|Y}\, p_{XY}$

**Def 17** (Quantum Markov chain)

A tripartite quantum state $\rho_{ABC}$ is called a quantum Markov chain $(A \to B \to C)$, if $\exists\, R: B \to BC$ s.t.

$$\rho_{ABC} = (id_A \otimes R)(\rho_{AB})$$

Informally: "Can recover $C$ from $B$"

**Thm 18** $\rho_{ABC}$ is a quantum Markov chain iff $I(A;C|B) = 0$.

Proof: $\Rightarrow$ Let $R: B \to BC$ be the quantum channel satisfying

$$\rho_{ABC} = (id_A \otimes R)(\rho_{AB}). \quad \Rightarrow \rho_{BC} = R(\rho_B)$$

$$D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) \underset{tr_C}{\overset{DPI}{\geq}} D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

$$\underset{id_A \otimes R}{\overset{DPI}{\geq}} D(\rho_{ABC} \| \rho_A \otimes \rho_{BC})$$

$$\Rightarrow D(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) = D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

$$-S(ABC) + S(A) + S(BC) = -S(AB) + S(A) + S(B)$$

$$\Leftrightarrow I(A;C|B) = 0$$

For the implication "$\Leftarrow$" need to study conditions for equality in data-processing

Recall: $I(A;C|B)=0 \Leftrightarrow \underline{D(\rho_{ABC} \| \rho_A \otimes \rho_{BC})} = D(\rho_{AB} \| \underline{\rho_A \otimes \rho_B})$

equality wrt. $\mathrm{tr}_C$

$\boxed{\text{Prop 19}}$ $D(\rho_{AB} \| \sigma_{AB}) = D(\rho_A \| \sigma_A)$ iff

there exists a recovery channel $R_\sigma : A \to AB$,

$$\underline{R_\sigma(X_A)} = \sigma_{AB}^{1/2} \left(\sigma_A^{-1/2} X_A \sigma_A^{-1/2} \otimes \mathbb{1}_B\right) \sigma_{AB}^{1/2}$$

($R_\sigma(\sigma_A) = \sigma_{AB}$ by definition), such that $R_\sigma(\rho_A) = \rho_{AB}$.
if $\operatorname{supp} \sigma_A = \mathcal{H}_A$

Proof: $\Leftarrow$ We first show that $R: A \to AB$ is a quantum channel

.) $R_\sigma$ is CP:

$$\left.\begin{array}{l} X_A \mapsto \sigma_A^{-1/2} X_A \sigma_A^{-1/2} \\ X_A \mapsto X_A \otimes \mathbb{1}_B \\ X_{AB} \mapsto \sigma_{AB}^{1/2} X_{AB} \sigma_{AB}^{1/2} \end{array}\right\} \text{CP} \Rightarrow R_\sigma \text{ is CP}$$

as a composition of CP maps.

·) $R_\sigma$ is TP on supp $\sigma_A$ :

$$\operatorname{tr}\left(R_\sigma(X_A)\right) = \operatorname{tr}\left(\sigma_{AB}^{1/2}\left(\sigma_A^{-1/2} X_A \sigma_A^{-1/2} \otimes \mathbb{1}_B\right)\sigma_{AB}^{1/2}\right)$$

$$= \operatorname{tr}\left(\sigma_{AB}\left(\sigma_A^{-1/2} X_A \sigma_A^{-1/2} \otimes \mathbb{1}_B\right)\right)$$

$$= \operatorname{tr}\left(\sigma_A \sigma_A^{-1/2} X_A \sigma_A^{-1/2}\right)$$

$$= \operatorname{tr}\left(\underbrace{\sigma_A^{-1/2}\sigma_A\sigma_A^{-1/2}}_{} X_A\right) = \operatorname{tr} X_A \quad \text{if}$$

$$\Pi X_A \Pi = X_A$$

$$\sigma_A^0 = \lim_{\alpha \to 0} \sigma_A^\alpha = \Pi_{\operatorname{supp}\sigma} \equiv \Pi$$

$\Rightarrow$ If $R_\sigma(\rho_A) = \rho_{AB}$ $\left(R_\sigma(\sigma_A) = \sigma_{AB}\right)$, then

$$D(\rho_{AB} \| \sigma_{AB}) \underset{\substack{\uparrow \\ \text{DPI wrt. } \operatorname{tr}_B}}{\geq} D(\rho_A \| \sigma_A) \underset{\substack{\uparrow \\ \text{DPI wrt. } R_\sigma}}{\geq} D(\rho_{AB} \| \sigma_{AB})$$