$\boxed{\text{Recap}}$

·) Relative entropy: ρ quantum state, $\sigma \geq 0$

$$D(\rho\|\sigma) = \begin{cases} \operatorname{tr} \rho(\log\rho - \log\sigma) & \text{if } \operatorname{supp}\rho \subseteq \operatorname{supp}\sigma \\ \infty & \text{else} \end{cases}$$

·) Data-processing inequality: $D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$

for any quantum channel $\mathcal{N}$

·) Functions on operators: A Hermitian with spectral decomp. $A = \sum_i \lambda_i |i\rangle\langle i|$

$f : I \to \mathbb{R}$ with $I \supseteq \operatorname{spec} A$: $\quad f(A) = \sum_i f(\lambda_i) |i\rangle\langle i|$

·) Examples: matrix logarithm $\quad \log\rho = \sum_{i:\lambda_i>0} (\log\lambda_i) |i\rangle\langle i|$

entropy function $\eta(t) = t\log t$: $\quad \eta(\rho) = \sum_i \eta(\lambda_i) |i\rangle\langle i|$

defined for all $\rho \geq 0$.

·) f is operator-convex if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$

$\lambda \in [0,1]$

·) Not every convex function is operator-convex (e.g., $t \mapsto t^3$)

·) $-\log t$ and $\eta(t) = t\log t$ <u>are</u> operator-convex.

·) Operator Jensen inequality: f operator-convex, V isometry:

$$f(V^\dagger A V) \leq V^\dagger f(A) V$$

·) Multiplication maps: $L_A(X) = AX$, $R_B(X) = XB$

·) $[L_A, R_B] = 0$

·) A Hermitian, $f: I \to \mathbb{R}$ with $\operatorname{spec} A \subseteq I$:

$$f(L_A) = L_{f(A)}, \quad f(R_A) = R_{f(A)}$$

$\boxed{\text{Def 8}}$ Let $X, Y \in B(\mathcal{H})$ be Hermitian, Y invertible.

The relative modular operator $\Delta \equiv \Delta^{X,Y}$ is defined as

$$\Delta^{X,Y} = L_X R_{Y^{-1}} : Z \mapsto X Z Y^{-1}$$

$\boxed{\text{Lemma 9}}$ Let $X, Y \geq 0$, Y invertible, and $\eta(t) = t \log t$.

Then $\eta(\Delta^{X,Y}) = \Delta^{X,Y} (L_{\log X} - R_{\log Y})$.

Proof: $X = \sum_i x_i |e_i\rangle\langle e_i|$, $Y = \sum_i y_i |f_i\rangle\langle f_i|$ (Y invertible $\Rightarrow$ $y_i > 0$)

spectral decompositions.

$$\Delta^{X,Y}(|e_i\rangle\langle f_j|) = L_X R_{Y^{-1}} (|e_i\rangle\langle f_j|) = X |e_i\rangle\langle f_j| Y^{-1}$$

$$= x_i |e_i\rangle\langle f_j| y_j^{-1} = x_i y_j^{-1} |e_i\rangle\langle f_j|$$

$\Rightarrow$ d^2 eigenmatrices: $|e_i\rangle\langle f_j|$ is an eigenmat. of $\Delta^{X,Y}$ with

eigenvalue $x_i y_j^{-1}$.

$$\eta(\Delta)(|e_i\rangle\langle f_j|) = \eta(x_i y_j^{-1})\, |e_i\rangle\langle f_j|$$

$\downarrow$

$\eta(t) = t\log t$

$$= x_i y_j^{-1}(\log x_i - \log y_j)\, |e_i\rangle\langle f_j|$$

$$\eta(\Delta) \overset{!}{=} \underline{\Delta^{x,y}(L_{\log x} - R_{\log y})}$$

$$\Delta^{x,y}(L_{\log x} - R_{\log y})(|e_i\rangle\langle f_j|) = x_i y_j^{-1}(\log x_i - \log y_j)\, |e_i\rangle\langle f_j|$$

Since $\{|e_i\rangle\langle f_j|\}_{i,j=1}^{d}$ forms a basis for $\mathcal{B}(\mathcal{H})$,

$$\eta(\Delta^{x,y}) = \Delta^{x,y}(L_{\log x} - R_{\log y}) \qquad \square$$

Now we're ready to prove data-processing for $D(\cdot\|\cdot)$ (Thm 5)

Proof of Thm 5: $\qquad$ $\operatorname{supp}\rho \not\subseteq \operatorname{supp}\sigma \Rightarrow D(\rho\|\sigma) \text{"="} \infty$

1) w.l.o.g. we assume $\operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$ $\Rightarrow$ σ can be taken to be invertible.

($\mathcal{H} = \ker\sigma \oplus \operatorname{supp}\sigma$; restrict $\mathcal{H}$ to $\operatorname{supp}\sigma$ by proj.)

2) remember: for any quantum channel $\mathcal{N}: A \to B$,

$\exists$ isometry $V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ s.t.

$$\mathcal{N}(X_A) = \operatorname{tr}_E V X_A V^\dagger$$

Recall: $f(VAV^\dagger) = V f(A) V^\dagger$ by definition

applying to matrix logarithm $\Rightarrow D(V\rho V^\dagger \| V\sigma V^\dagger) = D(\rho \| \sigma)$

$\mathcal{N} = \mathrm{tr}_E\, V \cdot V^\dagger$ $\Rightarrow$ claim follows from proving DPI

for $\mathcal{N} = \mathrm{tr}_B : AB \to A$

3) to show: $D(\rho_{AB} \| \sigma_{AB}) \geq D(\rho_A \| \sigma_A)$ $\quad$ (ρ_{AB} state, $\sigma_{AB} \geq 0$)

since $\operatorname{supp} \sigma_{AB} \subseteq \operatorname{supp} \sigma_A \otimes \operatorname{supp} \sigma_B$ $\quad$ (*)

w.l.o.g. assume both σ_{AB} and σ_A to be invertible.

((*) proved in arXiv: quant-ph/0512258, Lem. B.4.1)

Define $\Delta_{AB} = L_{\rho_{AB}} R_{\sigma_{AB}^{-1}}$, $\quad \Delta_A = L_{\rho_A} R_{\sigma_A^{-1}}$

$$\boxed{\langle X, Y \rangle = \mathrm{tr}\, X^\dagger Y \text{ on } B(\mathcal{H})}$$

$D(\rho_{AB} \| \sigma_{AB}) = \mathrm{tr}\, \rho_{AB} (\log \rho_{AB} - \log \sigma_{AB})$

$\eta(t) = t \log t$ $\quad = \langle \sigma_{AB}^{1/2}, \eta(\Delta_{AB})(\sigma_{AB}^{1/2}) \rangle$

$\overset{\text{lemma 9}}{=} \langle \sigma_{AB}^{1/2}, \Delta_{AB} (L_{\log \rho_{AB}} - R_{\log \sigma_{AB}})(\sigma_{AB}^{1/2}) \rangle$

$= \langle \sigma_{AB}^{1/2}, \rho_{AB} \log \rho_{AB}\, \sigma_{AB}^{1/2} \sigma_{AB}^{-1} - \rho_{AB}\, \sigma_{AB}^{1/2} \log \sigma_{AB}\, \sigma_{AB}^{-1} \rangle$

$= \mathrm{tr}\, \rho_{AB} \log \rho_{AB} - \mathrm{tr}\, \rho_{AB} \log \sigma_{AB}$

$D(\rho_A \| \sigma_A) = \langle \sigma_A^{1/2}, \eta(\Delta_A)(\sigma_A^{1/2}) \rangle$

$$D(\rho_{AB} \| \sigma_{AB}) = \langle \underline{\sigma_{AB}}^{1/2}, \eta(\underline{\Delta_{AB}})(\sigma_{AB}^{1/2}) \rangle$$

$$\geq$$

$$D(\rho_A \| \sigma_A) = \langle \underline{\sigma_A}^{1/2}, \eta(\Delta_A)(\sigma_A^{1/2}) \rangle$$

operator Jensen's inequality: V isometry: $f(V^\dagger X V) \leq V^\dagger f(X) V$

Goal: find $V: B(\mathcal{H}_A) \to B(\mathcal{H}_{AB})$ s.t.

i) V is an isometry: $V^\dagger V = id_A$

ii) $V^\dagger \Delta_{AB} V = \Delta_A$

iii) $V(\sigma_A^{1/2}) = \sigma_{AB}^{1/2}$

assume we have found such a V:

$$D(\rho_A \| \sigma_A) \overset{\text{Lemma 9}}{=} \langle \sigma_A^{1/2}, \eta(\Delta_A)(\sigma_A^{1/2}) \rangle$$

$$\overset{(ii)}{=} \langle \sigma_A^{1/2}, \eta(V^\dagger \Delta_{AB} V)(\sigma_A^{1/2}) \rangle$$

$$\underset{(i)}{\overset{\text{op. Jensen,}}{\leq}} \langle \underbrace{\sigma_A^{1/2}, V^\dagger}_{V(\sigma_A^{1/2})^\dagger} \eta(\Delta_{AB}) V(\sigma_A^{1/2}) \rangle$$

$$\overset{(iii)}{=} \langle \sigma_{AB}^{1/2}, \eta(\Delta_{AB})(\sigma_{AB}^{1/2}) \rangle \overset{\text{Lemma 9}}{=} D(\rho_{AB} \| \sigma_{AB})$$

Choice for isometry: $V(X_A) = (X_A \sigma_A^{-1/2} \otimes \mathbb{1}_B)\, \sigma_{AB}^{1/2}$

$$V : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_{AB})$$

i) easy to check: $V^\dagger(Y_{AB}) = \mathrm{tr}_B\left(Y_{AB}\, \sigma_{AB}^{1/2} (\sigma_A^{-1/2} \otimes \mathbb{1}_B)\right)$

$$V^\dagger V(X_A) = V^\dagger\left(X_A \sigma_A^{-1/2} \sigma_{AB}^{1/2}\right) \quad \text{(omit identity operators)}$$

$$= \mathrm{tr}_B\left(X_A \sigma_A^{-1/2} \underbrace{\sigma_{AB}^{1/2} \sigma_{AB}^{1/2}}_{\sigma_{AB}} \sigma_A^{-1/2}\right)$$

$$\xrightarrow{\mathrm{tr}_B}$$

$$= X_A \sigma_A^{-1/2} \sigma_A \sigma_A^{-1/2} = X_A \quad \forall X_A \Rightarrow V^\dagger V = \mathrm{id}_A$$

ii) $V^\dagger \Delta_{AB} V = \Delta_A$

$$\underline{V^\dagger \Delta_{AB} V}\,(X_A) = V^\dagger \Delta_{AB}\left(X_A \sigma_A^{-1/2} \sigma_{AB}^{1/2}\right)$$

$$= V^\dagger\left(\rho_{AB} X_A \sigma_A^{-1/2} \underbrace{\sigma_{AB}^{1/2} \sigma_{AB}^{-1}}_{\sigma_{AB}^{-1/2}}\right)$$

$$= \mathrm{tr}_B\left(\rho_{AB} X_A \sigma_A^{-1/2} \cancel{\sigma_{AB}^{-1/2}}\, \cancel{\sigma_{AB}^{1/2}}\, \sigma_A^{-1/2}\right)$$

$$= \mathrm{tr}_B\left(\rho_{AB} X_A \sigma_A^{-1}\right) = \rho_A X_A \sigma_A^{-1} = \Delta_A(X_A)$$

$$\forall X_A$$

iii) $V(\sigma_A^{1/2}) = \sigma_{AB}^{1/2}$ ✓

$\square$

Remarks: We proved DPI for quantum channels → generalize?

·) Dénes Petz: Proofs of DPI based on relative modular operators work for trace-preserving, 2-positive maps:

Φ is 2-positive if $\phi \otimes id_2$ is positive $\left(X_{AB} \geq 0 \Rightarrow (\phi \otimes id_2)(X_{AB}) \geq 0 \right)$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0 \quad \Rightarrow \quad \begin{pmatrix} \phi(A) & \phi(B) \\ \phi(C) & \phi(D) \end{pmatrix} \geq 0$$

block matrix
$A, B, C, D \in \mathcal{B}(\mathcal{H})$

Needed for these proofs: Schwarz inequality $\phi^\dagger(X^\dagger X) \geq \phi^\dagger(X^\dagger)\, \phi^\dagger(X)$

(holds for ϕ 2-positive and TP)

·) Very recent result (Müller-Hermes, Reeb arXiv: 1512.06117):

DPI holds for all trace-preserving, positive maps

(different proof based on complex interpolation)