Recap

·) Relative entropy: ρ quantum state, $\sigma \geq 0$

$$D(\rho \| \sigma) = \begin{cases} \operatorname{tr} \rho (\log \rho - \log \sigma) & \text{if } \operatorname{supp} \rho \subseteq \operatorname{supp} \sigma \\ \infty & \text{else} \end{cases}$$

·) Fundamental property: data-processing inequality

$\mathcal{N}$ quantum channel: $\quad D(\rho \| \sigma) \geq D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma))$

·) Some properties of $D(\cdot \| \cdot)$:

-) reduces to Kullback-Leibler divergence $D(p \| q) = \sum_x p_x \log \frac{p_x}{q_x}$

if ρ, σ are classical states ($\Longleftrightarrow [\rho, \sigma] = 0$)

-) for states ρ, σ: $D(\rho \| \sigma) \geq 0$ and $D(\rho \| \sigma) = 0$ iff $\rho = \sigma$.

-) Isometric invariance: $D(\rho \| \sigma) = D(V \rho V^\dagger \| V \sigma V^\dagger)$ for isometries V.

-) Classical-quantum states: if $\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x$, $\sigma_{XA} = \ldots$,

$$D(\rho_{XA} \| \sigma_{XA}) = \sum_x p_x \, D(\rho_A^x \| \sigma_A^x)$$

-) Joint convexity: $D\left(\sum_x \lambda_x \rho_x \middle\| \sum_x \lambda_x \sigma_x\right) \leq \sum_x \lambda_x D(\rho_x \| \sigma_x)$

where $\{\rho_x\}$ are states, $\{\sigma_x\}$ are PSD, and $\{\lambda_x\}$ is a P.D.

·) Functions on operators: Let A be Hermitian, $A = \sum_i \lambda_i |i\rangle\langle i|$ a spectral decomp.,

and $f: I \to \mathbb{R}$ with $\operatorname{spec} A \subseteq I$: $\quad f(A) := \sum_i f(\lambda_i) |i\rangle\langle i|$

Remarks on Prop 6:

i) Isometric invariance for $D(\cdot\|\cdot)$ can be proved directly (without DPI):

V isometry:

$$\log V\rho V^\dagger = V(\log\rho)V^\dagger \quad \text{by definition of functions on Hermitian op's}$$

$$D(V\rho V^\dagger \| V\sigma V^\dagger) = \operatorname{tr} V\rho V^\dagger \left(\log V\rho V^\dagger - \log V\sigma V^\dagger\right)$$

$$= \operatorname{tr} V\rho V^\dagger V\left(\log\rho - \log\sigma\right)V^\dagger$$

$$\overset{V^\dagger V = \mathbb{1}}{=} \operatorname{tr}\rho\left(\log\rho - \log\sigma\right) = D(\rho\|\sigma).$$

ii) Proving isometric invariance using DPI can be applied to every "divergence" ($:\Leftrightarrow$ satisfies data-processing)

iii) Joint convexity can be used to prove DPI (i.e, the two are equivalent)

Ex from QC1: $\forall \rho_{AB}$, $\int_{U_B} dU_B\, (\mathbb{1}_A\otimes U_B)\,\rho_{AB}\,(\mathbb{1}_A\otimes U_B)^\dagger = \rho_A \otimes \frac{1}{|B|}\mathbb{1}_B$ $\quad(*)$

to show: $D(\rho_{AB}\|\sigma_{AB}) \geq D(\rho_A\|\sigma_A)$ (together with isometric invariance, this proves data-processing)

$$D(\rho_{AB}\|\sigma_{AB}) = D(\underbrace{U_B}_{\mathbb{1}_A\otimes U_B}\rho_{AB}U_B^\dagger \| U_B\sigma_{AB}U_B^\dagger) \quad \text{by isometric invariance}$$

$$D(\rho_{AB}\|\sigma_{AB}) = \int_{U_B} dU_B\, D(U_B\rho_{AB}U_B^\dagger \| U_B\sigma_{AB}U_B^\dagger)$$

$$\geq D\left(\int dU_B\, U_B\rho_{AB}U_B^\dagger \,\Big\|\, \int dU_B\, U_B\sigma_{AB}U_B^\dagger\right)$$

$$\overset{(*)}{=} D\left(\rho_A\otimes\frac{1}{|B|}\mathbb{1}_B \,\Big\|\, \sigma_A\otimes\frac{1}{|B|}\mathbb{1}_B\right)$$

$$D\Big(\rho_A \otimes \underbrace{\tfrac{1}{|B|}\mathbb{1}_B}_{\pi_B} \,\Big\|\, \sigma_A \otimes \tfrac{1}{|B|}\mathbb{1}_B\Big) =$$

$$= \operatorname{tr}\Big[(\rho_A \otimes \pi_B)\big(\underbrace{\log(\rho_A \otimes \pi_B)}_{(\log \rho_A)\otimes \mathbb{1}_B \,-\, \log|B|\cdot \mathbb{1}_B} - \log(\sigma_A \otimes \pi_B)\big)\Big]$$

$$= \operatorname{tr}(\rho_A \otimes \pi_B)\big((\log \rho_A)\otimes \mathbb{1}_B - (\log \sigma_A)\otimes \mathbb{1}_B\big)$$

$$= \operatorname{tr} \rho_A \big(\log \rho_A - \log \sigma_A\big) = D(\rho_A \| \sigma_A)$$

$$\Rightarrow \quad D(\rho_{AB} \| \sigma_{AB}) \geq D(\rho_A \| \sigma_A) \quad \Rightarrow \text{DPI}$$

iv) In general: $D(\rho \otimes \omega \| \sigma \otimes \tau) = D(\rho\|\sigma) + D(\omega\|\tau)$ (Ex)

Functions on operators:

·) A Hermitian with spectral decomp. $A = \sum_i \lambda_i |i\rangle\langle i|$

$f: I \to \mathbb{R}$, $I \supseteq \operatorname{spec} A$: $\underline{f(A)} = \sum_i f(\lambda_i) |i\rangle\langle i|$

$$[A, f(A)] = 0$$

·) V isometry: $f(VAV^\dagger) = V f(A) V^\dagger$

·) Recall: partial order "$\leq$" on Hermitian operators:

$$A \leq B \;:\Leftrightarrow\; B - A \geq 0$$

$\to$ study functions that are operator convex: $\forall \lambda \in [0,1]$,

$$f(\lambda A + (1-\lambda) B) \leq \lambda f(A) + (1-\lambda) f(B)$$

·) Clear: every operator convex function is convex as a real fct.

(choose $\dim \mathcal{H} = 1$)

·) Converse is <u>not</u> true: e.g. $t \mapsto t^3$ is <u>not</u> op. convex (Ex)

·) Some examples of operator convex functions (no proofs for the time being)

Loewner-Heinz-Thm:

-) $t \mapsto t^p$ for $-1 \leq p \leq 0$ and $1 \leq p \leq 2$

-) $t \mapsto -t^p$ for $0 \leq p \leq 1$

-) $t \mapsto -\log t$

-) $t \mapsto \eta(t) = t \log t$ $\qquad \eta(0) = 0$

<u>Operator Jensen inequality</u> (proof later)

Let $f: I \to \mathbb{R}$ be operator convex, $V: \mathcal{H} \to \mathcal{K}$ an isometry,

$A \in \mathcal{B}(\mathcal{K})$ Hermitian with $\operatorname{spec} A \subseteq I$, then:

$$f(V^\dagger A V) \leq V^\dagger f(A) V.$$

<u>Detour 2:</u> Relative modular operator

Fix $A, B \in \mathcal{B}(\mathcal{H})$, define maps $L_A, R_B : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$:

$$L_A(X) = AX \quad , \quad R_B(X) = XB$$

$\boxed{\text{Lemma 7}}$ i) $[L_A, R_B] = 0$

ii) If A is invertible, then L_A, R_A are invertible, and $L_A^{-1} = L_{A^{-1}}, R_A^{-1} = R_{A^{-1}}$.

iii) If A is Hermitian, then so are L_A, R_A w.r.t. Hilbert-Schmidt inner product $\langle X, Y \rangle = tr(X^\dagger Y)$

iv) If A is Hermitian and $f: I \to \mathbb{R}$ with $\operatorname{spec} A \subseteq I$:

Then $f(L_A), f(R_A)$ are well-defined, and

$$f(L_A) = L_{f(A)}, \quad f(R_A) = R_{f(A)}.$$

<u>Proof</u>: i), ii) straightforward.

iii) $\langle X, L_A(Y) \rangle = \langle L_A^\dagger(X), Y \rangle$

$$\langle X, L_A(Y) \rangle = tr\, X^\dagger A Y = tr\left((A^\dagger X)^\dagger Y\right) = \langle A^\dagger X, Y \rangle$$

$$\Rightarrow L_A^\dagger = L_{A^\dagger} \overset{A = A^\dagger}{\Rightarrow} L_A^\dagger = L_A. \quad (\text{analogously for } R_A).$$

iv) Let $A = \sum_{i=1}^{d} a_i |i\rangle\langle i|$ $\quad d = \dim \mathcal{H}$

$$L_A(|i\rangle\langle j|) = A|i\rangle\langle j| = a_i |i\rangle\langle j|$$

d eigenoperators $|i\rangle\langle j|$ with eigenvalue a_i (for $j = 1, \dots, d$)

$\Rightarrow d^2$ eigenvalues with orthogonal eigenoperators $|i\rangle\langle j|$:

$$\langle |i\rangle\langle j|, |k\rangle\langle l| \rangle = \delta_{ik} \delta_{jl}$$

$\dim \mathcal{B}(\mathcal{H}) = d^2$

Define $f(L_A)$ through $f(L_A)(|i\rangle\langle j|) = f(a_i)|i\rangle\langle j| = L_{f(A)}(|i\rangle\langle j|)$

Since $\{|i\rangle\langle j|\}_{i,j=1}^{d}$ is a basis, $f(L_A) = L_{f(A)}$, and similarly for R_A. $\square$

$\boxed{\text{Def } 8}$ Let $X, Y \in \mathcal{B}(\mathcal{H})$ be Hermitian with Y invertible.

The relative modular operator $\Delta = \Delta^{X,Y}$ is defined as

$$\Delta^{X,Y} = L_X R_{Y^{-1}} \; : \; \Delta^{X,Y}(Z) = X Z Y^{-1}.$$