MATH 595 Quantum channels I: Representations and properties.

Exercise sheet 3 – July 22, 2021

Unless stated otherwise, $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \dots$ denote finite-dimensional Hilbert spaces.

1. Show that the amplitude damping channel $\mathcal{A}_\gamma$ with Kraus operators

$$K_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1| \qquad\qquad K_1 = \sqrt{\gamma}|0\rangle\langle 1|$$

   is covariant with respect to the group $G = \{\mathbb{1}, Z\}$ (and identical representations on the input and output space).

2. The von Neumann entropy S is defined for quantum states $\rho \in \mathcal{B}(\mathcal{H}), \rho \geq 0, \operatorname{tr}\rho = 1$ as $S(\rho) = -\operatorname{tr}\rho\log\rho$.[1] One of its properties is *subadditivity*:[2]

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \tag{1}$$

   (a) Let $\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes \rho_A^x$ be a classical-quantum state. Prove that

$$S(\rho_{XA}) = H(\{p_x\}) + \sum_x p_x S(\rho_A^x), \tag{2}$$

   where $H(\{p_x\}) = -\sum_x p_x \log p_x$ is the Shannon entropy.

   (b) Use (1) and (2) to prove concavity of the von Neumann entropy:

$$S\Big(\sum_x p_x\rho_x\Big) \geq \sum_x p_x S(\rho_x).$$

   Hint: Assemble the state ensemble $\{p_x, \rho_x\}$ in a classical-quantum state ρ_{XA} and evaluate both sides of (1) for this state.

3. In the lecture we encountered the *entanglement fidelity* of a quantum channel: Let $|\Phi^+\rangle_{AA'} = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle_A \otimes |i\rangle_{A'}$ with $d = \dim\mathcal{H}_A = \dim\mathcal{H}_{A'}$ be a maximally entangled state, and let $\mathcal{N}\colon A \to A$ be a quantum channel. Then the entanglement fidelity $F(\mathcal{N})$ is defined as

$$F(\mathcal{N}) = \langle\Phi^+|(\operatorname{id}_A \otimes \mathcal{N})(\Phi^+)|\Phi^+\rangle.$$

[1] Let $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ be a spectral decomposition of the state ρ. Then the operator $\log\rho$ is defined via spectral calculus:

$$\log\rho = \sum_{i\colon \lambda_i > 0} \log(\lambda_i)|\psi_i\rangle\langle\psi_i|.$$

[2] We will prove a stronger form of this property called *strong subadditivity* in MATH 595: Quantum channels II.

There is a related quantity called the *average fidelity*: let $\int_{\mathcal{U}(d)} dU$ denote the Haar integral with respect to the Haar measure on the unitary group $\mathcal{U}(d)$ on $\mathcal{H}_A$, and for a channel $\mathcal{N}: A \to B$ define the average fidelity $f(\mathcal{N})$ as

$$f(\mathcal{N}) = \int_{\mathcal{U}(d)} dU \langle\phi|U^\dagger \mathcal{N}(U\phi U^\dagger)U|\phi\rangle,$$

where $|\phi\rangle \in \mathcal{H}_A$ is a fixed but arbitrary pure state.

(a) Show that both $f(\mathcal{N})$ and $F(\mathcal{N})$ are invariant under channel twirling, i.e., $f(\mathcal{N}) = f(\mathcal{N}_{\mathcal{U}(d)})$ and $F(\mathcal{N}) = F(\mathcal{N}_{\mathcal{U}(d)})$ where $\mathcal{N}_{\mathcal{U}(d)}(X) = \int_{\mathcal{U}(d)} dU\, U^\dagger \mathcal{N}(UXU^\dagger)U$.

(b) Verify that a depolarizing channel $\mathcal{D}_q: \rho \mapsto (1-q)\rho + q\,\mathrm{tr}(\rho)\frac{1}{d}\mathbb{1}_A$ satisfies the following identity for entanglement fidelity and average fidelity:

$$f(\mathcal{D}_q) = \frac{F(\mathcal{D}_q)d + 1}{d+1}. \tag{3}$$

(c) Use Exercises 3a and 3b to show that (3) holds for *arbitrary* channels $\mathcal{N}$:

$$f(\mathcal{N}) = \frac{F(\mathcal{N})d + 1}{d+1}.$$

Hint: What do we know about the channel $\mathcal{N}_{\mathcal{U}(d)}$ in 3a?

4. Let G be a (finite or compact) group and $g \mapsto U_g$ be an irreducible unitary representation on $\mathcal{H}_A$. Show that, for any $X_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$,

$$\frac{1}{|G|}\sum_{g\in G}(U_g \otimes \mathbb{1}_B)X_{AB}(U_g \otimes \mathbb{1}_B)^\dagger = \frac{1}{|A|}\mathbb{1}_A \otimes X_B,$$

where $X_B = \mathrm{tr}_A X_{AB}$.

5. Let $\mathcal{H}$ be a Hilbert space of dimension d with basis $\mathfrak{B} = \{|i\rangle\}_{i=0}^{d-1}$ and consider the following two unitaries defined via their action on the basis $\mathfrak{B}$:

$$X|i\rangle = |i+1 \mod d\rangle \qquad\qquad Z|j\rangle = \omega^j|j\rangle,$$

where $\omega = \exp(2\pi i/d)$ is a primitive d-th root of unity. We have $X^d = Z^d = \mathbb{1}$. For $0 \le j,k \le d-1$, define the unitary $U_{j,k} = X^j Z^k$.

Prove the following statements:

(a) $U_{j,k}U_{l,m} = \omega^{kl}U_{j+l,k+m}$

(b) $\langle U_{j,k}, U_{l,m}\rangle = \mathrm{tr}(U_{j,k}^\dagger U_{l,m}) = d\delta_{jl}\delta_{km}$

(c) For $d = 2$, we have $(\sigma_X, \sigma_Y, \sigma_Z) = (U_{1,0}, iU_{1,1}, U_{0,1})$, where $(\sigma_X, \sigma_Y, \sigma_Z)$ are the usual Pauli matrices.

(d) The group $\langle U_{j,k} \colon 0 \leq j, k \leq d-1 \rangle$ is isomorphic to the discrete Heisenberg-Weyl group

$$\left\{ \begin{pmatrix} 1 & l & m \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} : k, l, m \in \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z} \right\}.$$