MATH 595 Quantum channels I: Representations and properties.

Exercise sheet 1 – July 22, 2021

Unless stated otherwise, $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2, \dots$ denote finite-dimensional Hilbert spaces.

1. Prove the following statements:

   (a) Let X be a Hermitian operator with largest eigenvalue λ_1. Then $X \leq \lambda_1 \mathbb{1}$. In particular, every quantum state ρ satisfies $\rho \leq \mathbb{1}$.
   Note: For Hermitian operators $A, B \in \mathcal{B}(\mathcal{H})$, the Löwner order *is defined as*

   $$A \geq B :\Leftrightarrow (A - B) \geq 0. \tag{1}$$

   (b) If $X, Y \geq 0$, then $X \otimes Y \geq 0$.

   (c) Let $A, B \in \mathcal{B}(\mathcal{H})$ be diagonalizable. Then A and B commute if and only if they are simultaneously diagonalizable: $[A, B] = AB - BA = 0$ iff there exists an invertible $S \in \mathcal{B}(\mathcal{H})$ such that both SAS^{-1} and SBS^{-1} are diagonal.

2. Prove the following "steering-like" identity: For every $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ there is a linear operator $K \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that

   $$|\psi\rangle = (\mathbb{1} \otimes K)|\gamma\rangle, \tag{2}$$

   where $|\gamma\rangle = \sum_{i=1}^{\dim \mathcal{H}_1} |i\rangle \otimes |i\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_1$ and $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_1}$ is an orthonormal basis for $\mathcal{H}_1$.

3. Let $\rho \in \mathcal{B}(\mathcal{H})$ be a quantum state, and let $|\psi^\rho\rangle \in \mathcal{H} \otimes \mathcal{H}'$ and $|\phi^\rho\rangle \in \mathcal{H} \otimes \mathcal{H}''$ be two purifications of ρ with purifying Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}''$, i.e., $\operatorname{tr}_{\mathcal{H}'} \psi^\rho = \rho = \operatorname{tr}_{\mathcal{H}''} \phi^\rho$. Assuming without loss of generality that $\dim \mathcal{H}' \leq \dim \mathcal{H}''$, prove that there exists an isometry $V \colon \mathcal{H}' \to \mathcal{H}''$ such that $|\phi^\rho\rangle = (\mathbb{1} \otimes V)|\psi^\rho\rangle$.
   Hint: Use the Schmidt decomposition *for bipartite states, which states the following: For any $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ there are non-negative coefficients $\{\lambda_i\}_{i=1}^r$ (called* Schmidt coefficients*) and sets of orthonormal vectors $\{|\alpha_i\rangle\}_{i=1}^r \subset \mathcal{H}_1$ and $\{|\beta_i\rangle\}_{i=1}^r \subset \mathcal{H}_2$ (called* Schmidt vectors*) such that $|\psi\rangle = \sum_{i=1}^r \lambda_i |\alpha_i\rangle \otimes |\beta_i\rangle$. The marginals of the pure state ψ on $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by*

   $$\rho_1 = \operatorname{tr}_2 \psi = \sum_{i=1}^r \lambda_i^2 \, |\alpha_i\rangle\langle\alpha_i| \qquad\qquad \rho_2 = \operatorname{tr}_1 \psi = \sum_{i=1}^r \lambda_i^2 \, |\beta_i\rangle\langle\beta_i|, \tag{3}$$

   respectively, and these are spectral decompositions. In particular, $\operatorname{rk} \rho_1 = \operatorname{rk} \rho_2 = r$, which is called the Schmidt rank *of $|\psi\rangle$. A pure state is entangled iff its Schmidt rank is greater than 1.*

4. Consider the CP map $T\colon \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2), T(X) = \sum_{j=1}^{3} K_j X K_j^\dagger$, where

$$K_1 = \begin{pmatrix} \sqrt{2/3} & 0 \\ 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} \frac{1+\omega}{\sqrt{6}} & 0 \\ 0 & \frac{1-\omega}{\sqrt{6}} \end{pmatrix} \qquad K_3 = \begin{pmatrix} \frac{1+\omega^2}{\sqrt{6}} & 0 \\ 0 & \frac{1-\omega^2}{\sqrt{6}} \end{pmatrix}, \tag{4}$$

and $\omega = \exp(2\pi i/3)$ is a third root of unity.

(a) Verify that T is trace-preserving.

(b) What is the Kraus rank of T?

(c) Determine an orthogonal Kraus representation, i.e., operators $\{L_i\}$ with $T(X) = \sum_i L_i X L_i^\dagger$ and $\langle L_i, L_j \rangle = \operatorname{tr}(L_i^\dagger L_j) \propto \delta_{ij}$.

(d) What is the physical interpretation of the channel T?

5. Let $\lambda \in (0,1)$ and consider the isometry $V\colon \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ defined via

$$\begin{aligned} |0\rangle &\mapsto \sqrt{\lambda}|0\rangle \otimes |0\rangle + \sqrt{1-\lambda}|1\rangle \otimes |1\rangle \\ |1\rangle &\mapsto \sqrt{1-\lambda}|0\rangle \otimes |0\rangle - \sqrt{\lambda}|1\rangle \otimes |1\rangle. \end{aligned} \tag{5}$$

Complete V to a unitary $U \in \mathcal{U}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, that is, find U such that $V = U(\mathbb{1} \otimes |\phi\rangle)$ for a suitable state $|\phi\rangle \in \mathbb{C}^2$.

Notation: We denote by $\mathcal{U}(\mathcal{H})$ the unitary group on $\mathcal{H}$.

6. Verify the following identities:

(a) $\operatorname{vec}(|\psi\rangle\langle\phi|) = |\psi\rangle \otimes |\bar{\phi}\rangle$ for $|\psi\rangle, |\phi\rangle \in \mathcal{H}$.

(b) $\operatorname{vec}(AXB) = (A \otimes B^T)\operatorname{vec}(X)$ for $X \in \mathcal{B}(\mathcal{H}), A \in \mathcal{B}(\mathcal{H}, \mathcal{H}'), B \in \mathcal{B}(\mathcal{H}', \mathcal{H})$.

(c) $\langle X, Y \rangle_{\mathcal{B}(\mathcal{H})} = \langle \operatorname{vec} X, \operatorname{vec} Y \rangle_{\mathcal{H} \otimes \mathcal{H}}$ for $X, Y \in \mathcal{B}(\mathcal{H})$.

7. Let $T\colon A \to B$ be a quantum channel.

(a) Determine an explicit form of a complementary channel $T^c\colon A \to E$ in terms of a Kraus representation of T.

(b) Use (a) to show that any channel in the set $\{U_E\, T^c(\cdot)\, U_E^\dagger \colon U_E \in \mathcal{U}(\mathcal{H}_E)\}$ is complementary to T.

Notation: We use the shorthand $T\colon R \to S$ for $T\colon \mathcal{B}(\mathcal{H}_R) \to \mathcal{B}(\mathcal{H}_S)$.