Recap

·) Let $\mathcal{N}: A \to B$ be (G, U_g, V_g)-covariant and $g \mapsto V_g$ irreducible.

Then $\mathcal{N}\left(\frac{1}{|\mathcal{K}|} \mathbb{1}_{\mathcal{K}}\right) = \frac{1}{|K|} \mathbb{1}_K$. (Correction to last lecture)

If $|\mathcal{H}| = |K|$, then $\mathcal{N}$ is unital.

·) Holevo information: $\chi(\mathcal{N}) = \max\limits_{\{p_x, \rho_x\}} S\left(\sum_x p_x \mathcal{N}(\rho_x)\right) - \sum_x p_x S(\mathcal{N}(\rho_x))$

·) Maximization can be restricted to pure-state ensembles.

·) Alternative expression for χ: Consider classical-quantum state

$$\sigma_{XB} = \sum_x p_x |x\rangle\langle x|_X \otimes \mathcal{N}(\rho_A^x),$$

then $\chi(\mathcal{N}) = I(X;B) = S(\sigma_X) + S(\sigma_B) - S(\sigma_{XB})$

·) $I(A;B)$ is called the mutual information.

·) Let $\mathcal{N}: A \to B$ be (G, U_g, V_g)-covariant.

-) If $g \mapsto U_g$ on $\mathcal{H}_A$ is irreducible:

$$\chi(\mathcal{N}) = S\left(\mathcal{N}\left(\frac{1}{|A|} \mathbb{1}_A\right)\right) - \min_{|\psi\rangle_A} S(\mathcal{N}(\psi_A))$$

-) If furthermore $g \mapsto V_g$ on $\mathcal{H}_B$ is irreducible:

$$\chi(\mathcal{N}) = \log |B| - \min_{|\psi\rangle_A} S(\mathcal{N}(\psi_A))$$

$\boxed{\text{Prop 23}}$ Let $\mathcal{N}: B(\mathcal{H}) \to B(\mathcal{K})$ be a (G, U_g, V_g)-covariant with $g \mapsto V_g$ on $\mathcal{K}$ is irreducible.

Then $\mathcal{N}\left(\frac{1}{|\mathcal{H}|} \mathbb{1}_{\mathcal{H}}\right) = \frac{1}{|\mathcal{K}|} \mathbb{1}_{\mathcal{K}}$. If $|\mathcal{H}| = |\mathcal{K}|$, then $\mathcal{N}$ is unital.

Proof: $V_g \mathcal{N}(.) V_g^{+} = \mathcal{N}(U_g . U_g^{+}) \quad \forall g \in G$

$$\Rightarrow \mathcal{N}(\mathbb{1}_{\mathcal{H}}) = \mathcal{N}(U_g U_g^{+}) = V_g \mathcal{N}(\mathbb{1}_{\mathcal{H}}) V_g^{+} \quad \forall g \in G$$

$$\Rightarrow \mathcal{N}(\mathbb{1}_{\mathcal{H}}) = \frac{1}{|G|} \sum_{g \in G} V_g \mathcal{N}(\mathbb{1}) V_g^{+} \overset{\text{Lemma 22}}{\Rightarrow} \mathcal{N}(\mathbb{1}_{\mathcal{H}}) = \frac{\underset{\substack{\downarrow \\ \operatorname{tr} \mathbb{1}_{\mathcal{H}} = |\mathcal{H}|}}{\operatorname{tr}} \mathcal{N}(\mathbb{1})}{|\mathcal{K}|} \mathbb{1}_{\mathcal{K}}$$

$$\Rightarrow \mathcal{N}(\mathbb{1}_{\mathcal{H}}) = \frac{|\mathcal{H}|}{|\mathcal{K}|} \mathbb{1}_{\mathcal{K}}. \qquad \square$$

Example of a channel with properties as in Prop 23 and $|\mathcal{H}| \neq |\mathcal{K}|$:

$$\mathcal{N}(X) = \frac{\operatorname{tr}(X)}{|\mathcal{K}|} \mathbb{1}_{\mathcal{K}}$$

This is a channel for all choices of $|\mathcal{H}|, |\mathcal{K}|$, and <u>all</u> unitary representations $g \mapsto U_g$ on $\mathcal{H}$ and $g \mapsto V_g$ on $\mathcal{K}$ for <u>any</u> G.

§ 3.4 Minimum output entropy and classical capacity

Recall: Holevo information $\chi(\mathcal{N}) = \max_{\{p_x, \rho_x\}} S\left(\sum_x p_x \mathcal{N}(\rho_x)\right) - \sum_x p_x S(\mathcal{N}(\rho_x))$

·) HSW-theorem: $\chi(\mathcal{N})$ is the classical capacity of $\mathcal{N}$

<u>IF</u> only product encodings are allowed.

·) Full classical capacity (allowing entangled inputs):

$$C(\mathcal{N}) = \sup_{n \in \mathbb{N}} \frac{1}{n} \chi(\mathcal{N}^{\otimes n}) = \lim_{n \to \infty} \frac{1}{n} \chi(\mathcal{N}^{\otimes n})$$

$$\geq \chi(\mathcal{N})$$

<u>Big question:</u> $\exists \mathcal{N} : C(\mathcal{N}) > \chi(\mathcal{N})$?

$\Leftrightarrow$ $\exists n \in \mathbb{N}$ s.t. $\chi(\mathcal{N}^{\otimes n}) > n \chi(\mathcal{N})$?

(non-additivity of χ)

Shor '04: χ is additive iff

a) minimum output entropy $S_{min}(\mathcal{N}) = \min_{|\psi\rangle} S(\mathcal{N}(\psi))$ is additive

iff b) entanglement of formation $E_F(\rho_{AB}) = \min_{\{p_x, |\psi_x\rangle\}} \sum_x p_x S(\mathrm{tr}_B \psi_x)$

where $\rho_{AB} = \sum_x p_x |\psi_x\rangle\langle\psi_x|$

is additive.

Construction of channels for which

(non-additivity of MOE)

$\Rightarrow$ (non-additivity of χ)

Let $N_1, N_2 : A_i \to B_i$ be such that

$$S_{min}(N_1 \otimes N_2) < S_{min}(N_1) + S_{min}(N_2)$$

$$N_i' : A_i C_i \to B_i, \quad N_i'(\rho \otimes \sigma) = \sum_{k,l} X^k Z^l N_i(\rho) (X^l Z^k)^\dagger \langle k,l|\sigma|k,l\rangle$$

$|C_i| = |B_i|^2$

X, Z ... generalized Pauli op's on B

($B_1 \cong B_2$)

$$\chi(N_i') = \max_{\{p_x, \rho_x\}} S\Big(\sum_x p_x N_i'(\rho_x)\Big) - \sum_x p_x S(N_i'(\rho_x))$$

$$\leq \log |B_i| - \underbrace{\min_{|\psi\rangle} S(N_i'(\psi))}_{S_{min}(N_i')}$$

$p_{k,l} = \dfrac{1}{|B_i|^2}$, $|\psi_{k,l}\rangle = |\psi\rangle \otimes |k,l\rangle$ where $|\psi\rangle$ achieves MOE.

$$\sum_{k,l} p_{k,l} N_i'(\psi \otimes |k,l\rangle\langle k,l|) = \frac{1}{|B_i|^2} \sum_{k,l} X^k Z^l N_i(\psi)(X^k Z^l)^\dagger \overset{\text{Lemma ??}}{=} \frac{1}{|B_i|} \mathbb{1}_{B_i}$$

$$S\Big(\sum_{h,l} p_{h,l}\, \mathcal{N}_i'(\varphi\otimes|h,l\rangle\langle h,l|)\Big) = \log|B_i|$$

$$S\big(\mathcal{N}_i'(\varphi\otimes|h,l\rangle\langle h,l|)\big) = S\big(X^h Z^l \mathcal{N}_i(\varphi)(X^h Z^l)^\dagger\big) \qquad \big[S(U\rho U^\dagger) = S(\rho)\big]$$

$$= S(\mathcal{N}_i(\varphi))$$

$$= S_{min}(\mathcal{N}_i).$$

$$\Rightarrow \boxed{\chi(\mathcal{N}_i') = \log|B_i| - S_{min}(\mathcal{N}_i)}$$

$$\chi(\mathcal{N}_1'\otimes\mathcal{N}_2') = \log|B_1| + \log|B_2| - \overbrace{S_{min}(\mathcal{N}_1\otimes\mathcal{N}_2)}^{< S_{min}(\mathcal{N}_1) + S_{min}(\mathcal{N}_2)}$$

$$> \log|B_1| + \log|B_2| - S_{min}(\mathcal{N}_1) - S_{min}(\mathcal{N}_2)$$

$$= \chi(\mathcal{N}_1') + \chi(\mathcal{N}_2').$$

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non-additivity of MOE $\Rightarrow$ non-additivity of χ

Classes of channels with strongly additive S_{min}: $S_{min}(\mathcal{N}\otimes\mathcal{M}) = S_{min}(\mathcal{N}) + S_{min}(\mathcal{M})$ for any $\mathcal{M}$

- ·) Entanglement-breaking channels (Shor '02)
- ·) d-dim depolarizing channels (King '03)
- ·) unital qubit channels (King '02)
- ·) Hadamard channels (King et al. '04)

Hastings ('08): Counter-example to additivity conjecture

Review of counter-example:

Open problem!

- based on random matrix theory $\rightarrow$ no explicit counter-example known.

- A, B quantum systems:

$$\mathcal{E}(\rho_A) = \mathrm{tr}_A\left(U_{AB}\,(\rho_A \otimes |0\rangle\langle 0|_B)\,U_{AB}^\dagger\right) \quad : A \rightarrow B$$

with environment A

$$\bar{\mathcal{E}}(\rho_A) = \mathrm{tr}_A\left(\bar{U}_{AB}\,(\rho_A \otimes |0\rangle\langle 0|_B)\,U_{AB}^T\right)$$

- Idea: $|A| \gg |B|$ and draw U_{AB} from Haar measure on $\mathcal{U}(|A|\cdot|B|)$

1) $\forall\, U \in \mathcal{U}(|A|\cdot|B|)$, $\quad S_{min}(\mathcal{E}\otimes\bar{\mathcal{E}}) < S_{min}(\mathcal{E}) + S_{min}(\bar{\mathcal{E}})$

$$S_{min}(\mathcal{E}\otimes\bar{\mathcal{E}}) \leq 2\log|B| - \frac{\log|B|}{|B|}$$

2) Let U be drawn from the Haar measure on $\mathcal{U}(|A|\cdot|B|)$

There exists c_0 s.t. for $c \geq c_0$, sufficiently large $|A|$,

$$\left(\frac{|B|}{|A|} = o(1)\right)$$

$$S_{min}(\mathcal{E}) = S_{min}(\bar{\mathcal{E}}) \geq \log|B| - \frac{c}{|B|}$$

with probability $1 - o(1)$.

3) If (2) is satisfied and $\log |B| > 2c$,

then $$S_{min}(\mathcal{E} \otimes \bar{\mathcal{E}}) \leq 2 \log |B| - \frac{\log |B|}{|B|}$$

$$< 2 \log |B| - \frac{2c}{|B|}$$

$$= S_{min}(\mathcal{E}) + S_{min}(\bar{\mathcal{E}})$$ "$\square$"