Recap

- $\mathcal{N}: A \to B$ channel, G group with unitary reps U_g on $\mathcal{H}_A$, V_g on $\mathcal{H}_B$

  $\mathcal{N}$ is called (G, U_g, V_g)-covariant if

$$V_g \mathcal{N}(.) V_g^\dagger = \mathcal{N}(U_g . U_g^\dagger) \quad \forall g \in G$$

- Examples:
  - Pauli channels $\rho \mapsto \underline{p_0} \rho + \underline{p_1} X \rho X + \underline{p_2} Y \rho Y + \underline{p_3} Z \rho Z$

    covariance group: Pauli group $\{\pm 1, \pm i\} \cup \{\mathbb{1}, X, Y, Z\}$
  - depolarizing channel $\rho \mapsto (1-q)\rho + q \, \mathrm{tr}(\rho) \frac{1}{d} \mathbb{1}$

    covariance group: $\mathcal{U}(d)$ (full unitary group)
  - erasure channel $\rho \mapsto (1-p)\rho + p \, \mathrm{tr}(\rho) |e\rangle\langle e|$

    covariance group: $\mathcal{U}(d)$

    representation on $\mathcal{H}_B$: $U \mapsto \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$

- $\mathcal{N}$ is (G, U_g, V_g)-covariant iff $\tau_{AB}^{\mathcal{N}} = (\overline{U}_g \otimes V_g) \tau_{AB}^{\mathcal{N}} (\overline{U}_g \otimes V_g)^\dagger$

  for all $g \in G$. $\quad (*)$

- This is basis-dependent due to basis choice for Choi operator.

- Basis-independent version: $J^{\mathcal{N}} := (\mathrm{id} \otimes \mathcal{N})(\mathbb{F}_{AA'})$ $\left(J^{\mathcal{N}} = (\tau^{\mathcal{N}})^{T_A} \right)$

  $(*)$ becomes $\quad J^{\mathcal{N}} = (U_g \otimes V_g) J^{\mathcal{N}} (U_g \otimes V_g)^\dagger$

$\boxed{\text{Prop 18}}$ Let $\mathcal{N}: A \to B$ a channel with $d = |A| = |B|$.

If $U\mathcal{N}(.)U^\dagger = \mathcal{N}(U.U^\dagger)$ $\forall\, U \in \mathcal{U}(d)$, then

$$\mathcal{N}(X) = (1-q)X + q\,\mathrm{tr}(X)\frac{1}{d}\mathbb{1}_d$$

where $q = \dfrac{f-d^2}{1-d^2}$ and $f = \langle\gamma|\tau^{\mathcal{N}}|\gamma\rangle$.

<u>Proof:</u> $\mathcal{N}$ is $(\mathcal{U}(d), \mathcal{U}, \mathcal{U})$-covariant $\underset{\text{remark}}{\overset{\text{Prop 17}}{\Longrightarrow}}$ $J^{\mathcal{N}} = (\mathrm{id}\otimes\mathcal{N})(\mathbb{F}_{AA'})$

satisfies $J^{\mathcal{N}} = (U\otimes U)J^{\mathcal{N}}(U\otimes U)^\dagger$ for all $U \in \mathcal{U}(d)$

<u>Claim (proof later):</u> $J^{\mathcal{N}} = x\,\mathbb{1}_A\otimes\mathbb{1}_B + y\,\mathbb{F}_{AB}$ $(*)$

$\mathbb{F}$ invariant under $U\otimes U$:

$\mathbb{F}(|\varphi\rangle\otimes|\psi\rangle) = |\psi\rangle\otimes|\varphi\rangle$

$(U\otimes U)\mathbb{F}(|\varphi\rangle\otimes|\psi\rangle) = U|\psi\rangle\otimes U|\varphi\rangle = \mathbb{F}(U|\varphi\rangle\otimes U|\psi\rangle)$

$\forall\, |\varphi\rangle, |\psi\rangle \Rightarrow [U\otimes U, \mathbb{F}] = 0$

$$\tau^{\mathcal{N}}_{AB} = \left(J^{\mathcal{N}}_{AB}\right)^{T_A} \overset{(*)}{=} x\,\underbrace{\mathbb{1}_A^T}_{=\mathbb{1}_A}\otimes\mathbb{1}_B + y\,\underbrace{\mathbb{F}_{AB}^{T_A}}_{=|\gamma\rangle\langle\gamma|_{AB}} = \underline{x\,\mathbb{1}_A\otimes\mathbb{1}_B + y\,|\gamma\rangle\langle\gamma|}$$

depolarizing channel: $X \mapsto (1-q)X + q\,\mathrm{tr}(X)\frac{1}{d}\mathbb{1}_d$

$\mathrm{tr}\,\tau^{\mathcal{N}}_{AB} = d$: $\quad d = x d^2 + y\overbrace{\langle\gamma|\gamma\rangle}^{d} \Rightarrow 1 = xd + y$

$f = \langle\gamma|\tau^{\mathcal{N}}_{AB}|\gamma\rangle$: $\quad f = x\langle\gamma|\mathbb{1}_{AB}|\gamma\rangle + y\,|\langle\gamma|\gamma\rangle|^2 = xd + yd^2$

$1 = xd + y$

$f = xd + yd^2 = 1 - y + yd^2 \Rightarrow y = \frac{1-f}{1-d^2}, \quad 1 - y = \frac{f-d^2}{1-d^2}$

$\tau_{AB}^N = x\, \mathbb{1}_A \otimes \mathbb{1}_B + y\, |\gamma\rangle\langle\gamma| . \qquad y = 1 - q \Rightarrow q = \frac{f-d^2}{1-d^2} \quad \square$

$\boxed{\text{Lemma 19}}$ Let $R \in B(\mathbb{C}^d \otimes \mathbb{C}^d)$ be such that

$$(U \otimes U) R (U \otimes U)^\dagger = R \quad \text{for all } U \in \mathcal{U}(d)$$

Then there are $a, b \in \mathbb{C}$ s.t. $R = a\, \mathbb{1}_1 \otimes \mathbb{1}_2 + b\, \mathbb{F}_{12}$.

Proof: .) Using Schur-Weyl-duality: $\mathcal{H} = (\mathbb{C}^d)^{\otimes 2}$

$$\varphi_{U_2}: \mathcal{U}(2) \to GL(\mathcal{H}), \qquad U \mapsto U \otimes U$$

$$(*)$$

$$\varphi_{S_2}: S_2 \to GL(\mathcal{H}), \qquad \pi \mapsto \left(|\psi_1\rangle \otimes |\psi_2\rangle \mapsto |\psi_{\pi^{-1}(1)}\rangle \otimes |\psi_{\pi^{-1}(2)}\rangle\right)$$

φ_{U_2} and φ_{S_2} commute with each other:

$$(U \otimes U) \mathbb{F}(|\psi_1\rangle \otimes |\psi_2\rangle) = \mathbb{F}(U \otimes U)(|\psi_1\rangle \otimes |\psi_2\rangle)$$

$$\Rightarrow \quad [U \otimes U, \mathbb{F}] = 0$$

Commutant of a subalgebra $\mathcal{A} \subseteq \mathcal{C}$: $\quad \mathcal{A}' = \{ b \in \mathcal{C} : ab = ba \ \forall a \in \mathcal{A} \}$

Define $\mathcal{A} = \langle \{\varphi_{U_2}(U)\}_{U \in \mathcal{U}_2} \rangle_{\mathbb{C}}, \quad \mathcal{B} = \langle \{\varphi_{S_2}(\pi)\}_{\pi \in S_2} \rangle_{\mathbb{C}} = \langle \{\mathbb{1}, \mathbb{F}\} \rangle_{\mathbb{C}}$

Schur-Weyl duality: $\mathcal{A}' = \mathcal{B}, \ \mathcal{B}' = \mathcal{A}$ (holds more generally for representations $(*)$ on $(\mathbb{C}^d)^{\otimes N}$)

$(U\otimes U)R(U\otimes U)^\dagger = R \Rightarrow R \in \mathcal{A}' = \mathcal{B} \Rightarrow \exists x,y \in \mathbb{C} : R = x\mathbb{1} + y\mathbb{F}.$

.) direct proof for $d=2$ (can be generalized to $d>2$):

to show: $(U\otimes U)R(U\otimes U)^\dagger = R$ for all $U \in \mathcal{U}(2)$

$$\Rightarrow R = x\,\mathbb{1}_2\otimes\mathbb{1}_2 + y\,\mathbb{F}_{22}$$

Bell basis:
$$\left.\begin{aligned} |\phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ \underline{|\psi^-\rangle} &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}\right\} \text{orthonormal basis for } \mathcal{H} = \mathbb{C}^2\otimes\mathbb{C}^2$$

define: symmetric subspace $\mathcal{H}_s = \{|\varphi\rangle \in \mathcal{H} : \mathbb{F}|\varphi\rangle = |\varphi\rangle\}$

antisymmetric subsp. $\mathcal{H}_a = \{|\varphi\rangle \in \mathcal{H} : \mathbb{F}|\varphi\rangle = -|\varphi\rangle\}$

easy to see: $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle \in \mathcal{H}_s$, $|\psi^-\rangle \in \mathcal{H}_a$

$$\mathcal{H}_s \cap \mathcal{H}_a = \{0\}$$

$$\Rightarrow \mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_a$$

define projectors $P_s = \frac{1}{2}(\mathbb{1} + \mathbb{F})$, $P_a = \frac{1}{2}(\mathbb{1} - \mathbb{F})$

$$\Rightarrow \mathcal{H}_{s,a} = P_{s,a}\mathcal{H}$$

Express R in Bell basis:

$$R = \begin{array}{c|c} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} & \begin{array}{c} \phi^+ \\ \phi^- \\ \psi^+ \\ \psi^- \end{array} \\ \begin{array}{cccc} \phi^+ & \phi^- & \psi^+ & \psi^- \end{array} & \end{array}$$

by assumption, $(U \otimes U) R (U \otimes U)^\dagger = R$

We now make special choices for U:

1) $X \otimes X$: $|\phi^+\rangle \mapsto |\phi^+\rangle$

$X|0\rangle = |1\rangle$ $\quad |\phi^-\rangle \mapsto -|\phi^-\rangle$

$X|1\rangle = |0\rangle$ $\quad |\psi^+\rangle \mapsto |\psi^+\rangle$

$|\psi^-\rangle \mapsto -|\psi^-\rangle$

$(X \otimes X)|\phi^-\rangle\langle\phi^+|(X \otimes X) = -|\phi^-\rangle\langle\phi^+|$

$$R = \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix}$$

2) $Z \otimes Z$: $|\phi^+\rangle \mapsto |\phi^+\rangle$

$Z|0\rangle = |0\rangle$ $\quad |\phi^-\rangle \mapsto |\phi^-\rangle$

$Z|1\rangle = -|1\rangle$ $\quad |\psi^+\rangle \mapsto -|\psi^+\rangle$

$|\psi^-\rangle \mapsto -|\psi^-\rangle$

$$R = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ 0 & & & b \end{pmatrix}$$

$a_i, b \in \mathbb{C}$

3) Hadamard unitary: $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$H \otimes H$: $|\phi^+\rangle \mapsto |\phi^+\rangle$

$|\phi^-\rangle \mapsto |\psi^+\rangle$

$|\psi^+\rangle \mapsto |\phi^-\rangle$

$|\psi^-\rangle \mapsto -|\psi^-\rangle$

$\Rightarrow a_2 = a_3$

4) Phase gate $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$$R = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_2 & \\ 0 & & & b \end{pmatrix}$$

$$\left.\begin{array}{l|l} S|0\rangle = |0\rangle & S \otimes S: \ |\phi^+\rangle \mapsto |\phi^-\rangle \\ S|1\rangle = i|1\rangle & \qquad\quad |\phi^-\rangle \mapsto |\phi^+\rangle \\ & \qquad\quad |\psi^+\rangle \mapsto i|\psi^+\rangle \\ & \qquad\quad |\psi^-\rangle \mapsto i|\psi^-\rangle \end{array}\right\} \quad a_1 = a_2 = a$$

$$\Rightarrow R = \begin{pmatrix} a & & & 0 \\ & a & & \\ & & a & \\ 0 & & & b \end{pmatrix} = a P_s + b P_a \qquad \left[P_{s/a} = \frac{1}{2}(\mathbb{1} \pm \mathbb{F}) \right]$$

$$= \frac{a+b}{2} \mathbb{1} + \frac{a-b}{2} \mathbb{F}$$

$\square$