Recap

·) Degradable channels: A channel $\mathcal{N}: A \to B$ with complementary channel $\mathcal{N}^c: A \to E$ is called degradable, if there is a degrading map $\mathcal{D}: B \to E$ s.t. $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$.

·) Intuition: Bob can locally simulate the environment

·) Quantum capacity of degradable channels is understood and efficiently computable.

·) Examples:
- erasure channel $\mathcal{E}_p$ for $p \leq \frac{1}{2}$
- amplitude damping channel $\mathcal{A}_\gamma$ for $\gamma \leq \frac{1}{2}$
- generalized dephasing channels
- complementary channels of entanglement-breaking ch.

·) Degradable channels do not form a convex set (antideg. channels do)

·) Every qubit-qubit channel with qubit environment is degradable or antidegradable.

·) A channel that is both degradable and antidegradable is called symmetric. (Ex.: 50-50 erasure channel)

$\boxed{\text{Def 16}}$ Let $\mathcal{N}: A \to B$ be a quantum channel and G be a group with unitary representations U_g on $\mathcal{H}_A$ and V_g on $\mathcal{H}_B$.

Then $\mathcal{N}$ is called covariant w.r.t. (G, U_g, V_g) if

$$V_g \mathcal{N}(.) V_g^\dagger = \mathcal{N}(U_g . U_g^\dagger) \text{ for all } g \in G.$$

Representation theory basics:

·) (φ, V) is a rep. of G: $\varphi: G \to GL(V)$, $\varphi(gh) = \varphi(g)\varphi(h)$

subspace $W \subseteq V$ is called G-invariant, if $\varphi(g) w \in W$ $\forall w \in W, \forall g \in G$

·) $\{0\}, V$ are always G-invariant

·) (φ, V) is called irreducible, if $\{0\}, V$ are the only G-inv. subsp.

·) Schur's lemma: $(\varphi, V), (\psi, W)$ rep's of G

G-linear map $f: V \to W$: $f \circ \varphi(g) = \psi(g) \circ f$

If φ, ψ are irreps, then either $V \ncong W$ and $f = 0$

or $V \cong W$ and $f = \lambda\, id_{V \to W}$ for some $\lambda \in \mathbb{C}$.

<u>Some examples:</u>

·) Pauli channels: $\rho \mapsto p_0 \rho + p_1 X \rho X + p_2 Y \rho Y + p_3 Z \rho Z$

covariance group: Pauli group $P = \{\pm 1, \pm i\} \cup \{\mathbb{1}, X, Y, Z\}$

$\theta_1, \theta_2 \in P: \quad \theta_1\theta_2 \cdot \theta_2^\dagger \theta_1^\dagger = \theta_2\theta_1 \cdot \theta_1^\dagger \theta_2^\dagger$

·) depolarizing channel: $\rho \mapsto (1-p)\rho + p/_3\,(X\rho X + Y\rho Y + Z\rho Z)$

covariance group: $\mathcal{U}(2)$ $\qquad \updownarrow\ q = \frac{4p}{3}$

easy to see in the "q-representation": $\underline{\rho \mapsto (1-q)\rho + q\,\mathrm{tr}(\rho)\,\frac{1}{2}\mathbb{1}}$

$$\mathcal{D}_p(U\rho U^\dagger) = (1-q)\,U\rho U^\dagger + q\,\mathrm{tr}(U\rho U^\dagger)\,\frac{1}{2}\underset{\underset{U\cdot U^\dagger}{\uparrow}}{\mathbb{1}}$$

$$= U\mathcal{D}_p(\rho)U^\dagger$$

in d dimensions $(d \geq 2)$: $\underline{\rho \mapsto (1-q)\rho + q\,\mathrm{tr}(\rho)\,\frac{1}{d}\mathbb{1}}$

covariance group: $\mathcal{U}(d)$

·) amplitude damping channel $A_\gamma = \{K_0, K_1\}$ $\qquad K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$

$\qquad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$

covariance group: $\{\mathbb{1}, Z\} \cong \mathbb{Z}_2$

·) erasure channel $\mathcal{E}_p(\rho) = (1-p)\rho + p\,\mathrm{tr}(\rho)\,|e\rangle\langle e|$

covariance group: $\mathcal{U}(2)$ $\qquad$ input space: $\mathcal{U}(2)$

output space: $U \mapsto \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$

in d dimensions $(d \geq 2)$: d-dimensional erasure with covariance group $\mathcal{U}(d)$

Prop 17 A channel $\mathcal{N}: A \to B$ is (G, U_g, V_g)-covariant

iff $\tau_{AB}^{\mathcal{N}} = (\bar{U}_g \otimes V_g)\, \tau_{AB}^{\mathcal{N}}\, (\bar{U}_g \otimes V_g)^\dagger \quad \forall g \in G.$

Proof: $\Rightarrow$ $\quad V_g \mathcal{N}(.) V_g^\dagger = \mathcal{N}(U_g . U_g^\dagger) \quad \forall g \in G$

$$\Leftrightarrow \quad \mathcal{N}(.) = V_g^\dagger \mathcal{N}(U_g . U_g^\dagger) V_g \quad \forall g \in G$$

$$\tau_{AB}^{\mathcal{N}} = (id \otimes \mathcal{N})(\gamma)$$

$$= (\mathbb{1}_1 \otimes V_g^\dagger)\, (id \otimes \mathcal{N}) \Big((\mathbb{1}_1 \otimes U_g) \gamma (\mathbb{1} \otimes U_g^\dagger) \Big) (\mathbb{1}_1 \otimes V_g)$$

[Transpose trick (ricochet trick): $(\mathbb{1} \otimes X) |\gamma\rangle = (X^T \otimes \mathbb{1}) |\gamma\rangle$

useful implication: $(U \otimes \bar{U}) |\gamma\rangle = (UU^\dagger \otimes \mathbb{1}) |\gamma\rangle = |\gamma\rangle$

for any unitary U.]

$$= (\mathbb{1} \otimes V_g^\dagger)\, (id \otimes \mathcal{N}) \Big((U_g^T \otimes \mathbb{1}_2) \gamma\, (U_g^T \otimes \mathbb{1})^\dagger \Big) (\mathbb{1} \otimes V_g)$$

$$= (U_g^T \otimes V_g^\dagger)\, \underbrace{(id \otimes \mathcal{N})(\gamma)}_{\tau_{AB}^{\mathcal{N}}}\, (\bar{U}_g \otimes V_g)$$

$$\boxed{\begin{array}{l} \cdot)\ (\bar{U})^{-1} = U^T \\ \cdot)\ (U_{g^{-1}})^T = (U_g^\dagger)^T = \bar{U}_g \end{array}}$$

$$\Rightarrow \quad \tau_{AB}^{\mathcal{N}} = (U_g^T \otimes V_g^\dagger)\, \tau_{AB}^{\mathcal{N}}\, (\bar{U}_g \otimes V_g) \qquad \forall g \in G$$

$g \to g^{-1}$: if φ is a unitary rep., $\varphi(g)^{-1} = \varphi(g)^\dagger$

$$\Rightarrow \quad \tau_{AB}^{\mathcal{N}} = (\bar{U}_g \otimes V_g)\, \tau_{AB}^{\mathcal{N}}\, (\bar{U}_g \otimes V_g)^\dagger \qquad \checkmark$$

$\Leftarrow$ $\quad \tau_{AB}^{N} = (\bar{U}_g \otimes V_g)\, \tau_{AB}^{N}\, (\bar{U}_g \otimes V_g)^\dagger \quad \forall g \in G$

$$\Leftrightarrow \quad (U_g^T \otimes \mathbb{1})\, \tau_{AB}^{N}\, (U_g^T \otimes \mathbb{1}_2)^\dagger = (\mathbb{1} \otimes V_g)\, \tau_{AB}^{N}\, (\mathbb{1} \otimes V_g)^\dagger$$

recall Choi isomorphism: $\mathcal{N}(X) = \mathrm{tr}_1\, \tau_{AB}^{N}\, (X^T \otimes \mathbb{1})$

$$\mathcal{N}(U_g X U_g^\dagger) = \mathrm{tr}_1 \left[\tau_{AB}^{N} \left(\underbrace{(U_g X U_g^\dagger)^T}_{\bar{U}_g X^T U_g^T} \otimes \mathbb{1} \right) \right]$$

cyclicity of tr_1

$$= \mathrm{tr}_1 \left[(U_g^T \otimes \mathbb{1})\, \tau_{AB}\, (U_g^T \otimes \mathbb{1})^\dagger\, (X^T \otimes \mathbb{1}) \right]$$

$$= \mathrm{tr}_1 \left[(\mathbb{1} \otimes V_g)\, \tau_{AB}\, (\mathbb{1} \otimes V_g)^\dagger\, (X^T \otimes \mathbb{1}) \right]$$

$$= V_g\, \underbrace{\mathrm{tr}_1 \left[\tau_{AB}\, (X^T \otimes \mathbb{1}) \right]}_{\mathcal{N}(X)}\, V_g^\dagger$$

$$= V_g\, \mathcal{N}(X)\, V_g^\dagger \qquad \square$$

Problem: Prop 1) is basis-dependent (via Choi op.)

Solution: basis-independent version based on the Jamiołkowski-op.

$$J_{AB}^{N} = (id \otimes \mathcal{N})(\mathbb{F}_{AA'}) \qquad \mathbb{F}_{AB} = |\gamma\rangle\langle\gamma|^{T_A}$$

J_{AB}^{N} is related to τ_{AB}^{N} via T_A: $\quad (J_{AB}^{N})^{T_A} = \tau_{AB}^{N}$

In terms of the Jamiołkowski-op., (G, U_g, V_g)-covariance of $\mathcal{N}$ is equivalent to $(U_g \otimes V_g)\, J_{AB}^{\mathcal{N}}\, (U_g \otimes V_g)^\dagger = J_{AB}^{\mathcal{N}} \quad \forall g \in G$.

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d-dimensional dep. channel $\rho \mapsto (1-q)\rho + q\, \mathrm{tr}(\rho) \frac{1}{d} \mathbb{1}_d$

has $\mathcal{U}(d)$ as covariance group.

Prop 18 Let $\mathcal{N}: A \to B$ be a channel with $|A| = |B| = d$.

If $U \mathcal{N}(.) U^\dagger = \mathcal{N}(U . U^\dagger)$ for all $U \in \mathcal{U}(d)$,

then $\mathcal{N} = (1-q) . + q\, \mathrm{tr}(.) \frac{1}{d} \mathbb{1}_d$

with $q = \dfrac{f - d^2}{1 - d^2}$ where $f = \langle \gamma | \tau^{\mathcal{N}} | \gamma \rangle$.