Recap

- Entanglement-breaking channel: $(\mathrm{id}_R \otimes \mathcal{N})(\rho_{RA}) \in SEP$ for all ρ_{RA}
- $\mathcal{N}$ EB $\Leftrightarrow$ Choi op $\tau^{\mathcal{N}}$ is separable

  $\Leftrightarrow$ there is a Kraus rep. $\{K_i\}$ with $\mathrm{rk}\, K_i = 1 \;\; \forall i$

  $\Leftrightarrow$ $\mathcal{N}$ is a measure-and-prepare channel:

  $\exists$ POVM $\{E_i\}$ and states $\{\sigma_i\}$ s.t. $\mathcal{N}(\rho) = \sum_i \mathrm{tr}(E_i \rho)\, \sigma_i$
- EB channels can never generate entanglement $\Rightarrow$ zero quantum cap.
- EB channels break entanglement between different inputs

  $\Rightarrow$ classical capacity $C(\mathcal{N}) = \chi(\mathcal{N})$ Holevo information
- However: computing $\chi(\mathcal{N})$ is NP-hard :-(
- In general, deciding separability is NP-hard

  $\Rightarrow$ need for easily computable separability criterion
- Positive partial transpose (PPT): $\rho_{AB}^{T_B} \geq 0$

  ρ_{AB} SEP $\Rightarrow$ ρ_{AB} PPT
- If $|A| \cdot |B| \leq 6$, also sufficient criterion: ρ_{AB} PPT $\Rightarrow$ ρ_{AB} SEP
- PPT-channels: $(\mathrm{id}_R \otimes \mathcal{N})(\rho_{RA})$ PPT for all ρ_{RA}
- $\mathcal{N}$ EB $\Rightarrow$ $\mathcal{N}$ PPT

Prop 12 TFAE:

a) $\mathcal{N}$ is PPT

b) Choi op $\tau^{\mathcal{N}}$ is PPT

c) $\vartheta \circ \mathcal{N}$ is CP

$\vartheta : X \mapsto X^T$

Proof: a) $\Rightarrow$ b) $(id_A \otimes \mathcal{N})(\rho_{AA'})$ is PPT, in particular for $\rho = \gamma_{AA'}$ ✓

b) $\Rightarrow$ c) $$(id_A \otimes \vartheta \circ \mathcal{N})(\gamma) = (id_A \otimes \vartheta)(id \otimes \mathcal{N})(\gamma_{AA'})$$
$$= (id_A \otimes \vartheta)(\tau^{\mathcal{N}}_{AB})$$
$$= \left(\tau^{\mathcal{N}}_{AB}\right)^{T_B} \geq 0 \quad ✓$$

c) $\Rightarrow$ a) $\vartheta \circ \mathcal{N}$ is CP: $(id_R \otimes \vartheta \circ \mathcal{N})(\rho_{RA}) \geq 0 \quad \forall \rho_{RA} \geq 0$

$\Rightarrow \mathcal{N}$ is PPT $\square$

Remark: In general, $\vartheta \circ \mathcal{N}$ is <u>not</u> CP.

However, for every CP map $\mathcal{N}$, the map $\vartheta \circ \mathcal{N} \circ \vartheta$ is CP. (Ex)

What about the capacities of PPT channels?

- Horodecki[3]: PPT states are undistillable. $\rho_{AB}^{\otimes n}$

Entanglement distillation: Given: iid copies of a state ρ_{AB}

Goal: convert $\rho_{AB}^{\otimes n}$ to a smaller number m_n of maximally entangled states $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$

If there is an entanglement distillation protocol (LOCC) such that i) error of the protocol goes to 0 as $n \to \infty$,

ii) rate $c := \lim_{n\to\infty} \frac{m_n}{n} > 0$,

then ρ_{AB} is distillable.

Horodecki[3] $\Rightarrow$ $Q(\mathcal{N}) = 0$ for PPT-channels $\mathcal{N}$

(holds even if two-way CC is allowed!)

Remarks:
- for quantum capacity $Q(\mathcal{N})$ we only allow forward / one-way CC.
- If you also want to consider 2-way CC, then the relevant capacity is the two-way Q. cap. Q_2 (or $Q_{\leftrightarrow}$)
- $Q(\mathcal{N}) \leq Q_2(\mathcal{N}) \quad \forall \mathcal{N}$
- $\mathcal{N}$ is PPT $\Rightarrow$ $Q(\mathcal{N}) = Q_2(\mathcal{N}) = 0$.

Classical capacity of PPT channels is generally unknown!

§2.5 Antidegradable channels

quantum channel $\mathcal{N}: A \to B$ with an isometry $V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$

s.t. $\mathcal{N}(\rho) = \mathrm{tr}_E V \rho V^\dagger$.

Complementary channels: $\mathcal{N}^c(\rho) = \mathrm{tr}_B V \rho V^\dagger$

$$\begin{tikzcd} & B \\ A \arrow[ur, "\mathcal{N}"] \arrow[dr, "\mathcal{N}^c"'] & \\ & E \arrow[uu, bend right, "\mathcal{A}"'] \end{tikzcd}$$

Def 13 $\mathcal{N}$ as above is called antidegradable, if there is a channel $\mathcal{A}: E \to B$ s.t.

$$\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$$

Intuition: Eve (environment) can locally obtain Bob's output via the channel $\mathcal{A}$.

Antideg. channels cannot transmit quantum information: $Q(\mathcal{N}) = 0$

"Proof": Assume $Q(\mathcal{N}) > 0 \Leftrightarrow$ Alice can faithfully send qubits to Bob at a positive rate. But there is a protocol based on the channel $\mathcal{A}$ that lets Eve implement the same protocol that Alice and Bob use.

This would clone Alice's state to $B/E \Rightarrow$ violates no-cloning!

↯. "$\square$"

Examples of antidegradable channels:

a) erasure channel $\mathcal{E}_p : \rho \mapsto (1-p)\rho + p \operatorname{tr}(\rho)\, |e\rangle\langle e|$ for $p \geq \frac{1}{2}$.

b) amplitude damping channel $\mathcal{A}_\gamma$ for $\gamma \geq \frac{1}{2}$

c) depolarizing channel $\mathcal{D}_p$ for $p \geq \frac{1}{4}$.