Recap

- Rank of Kraus operators is not constant.
- Every unital qubit channel is unitary equivalent to a Pauli channel.

  Ex. $\Rightarrow$ Every unital qubit channel is mixed unitary.
- generalized dephasing channels: $\{|i\rangle\}_{i=1}^{d}$ ONB, $\{|\varphi_i\rangle\}_{i=1}^{d}$ arbitrary

$$V: |i\rangle_A \mapsto |i\rangle_B \, |\varphi_i\rangle_E$$

- gen. dephasing channels have full classical capacity $C(\mathcal{N}) = \log d$.
- Holevo information:
  - quantum state ensemble $E = \{p_x, \rho_x\}$ $\{p_x\}_x$ prob. dist.
  - $\chi(E, \mathcal{N}) := S\left(\sum_x p_x \mathcal{N}(\rho_x)\right) - \sum_x p_x S(\mathcal{N}(\rho_x))$
  - $\chi(\mathcal{N}) := \max_E \chi(E, \mathcal{N})$
  - HSW-theorem: classical capacity C satisfies

$$C(\mathcal{N}) \geq \chi(\mathcal{N})$$

  - Full formula: $C(\mathcal{N}) = \sup_{n \in \mathbb{N}} \frac{1}{n} \chi(\mathcal{N}^{\otimes n})$
  - problem, unless $\chi(\mathcal{N}^{\otimes n}) = n\chi(\mathcal{N})$ (not true in general(!))
- Entanglement-breaking channels: $(\mathrm{id}_R \otimes \mathcal{N})(\rho_{RA})$ separable $\forall \rho_{RA}$.

$\boxed{\text{Prop 10}}$ TFAE:

a) $\mathcal{N}: A \to B$ is EB

b) $\tau_{AB}^{\mathcal{N}} = (id_A \otimes \mathcal{N})(\gamma_{AA'})$ is separable.

c) $\mathcal{N}$ has a Kraus representation $\{K_i\}$ with $\mathrm{rk}\, K_i = 1 \;\forall i$.

d) $\mathcal{N}$ is a measure-and-prepare channel:

$\exists$ POVM $E = \{E_i\}$ and states $\{\sigma_i\}$ s.t.

$$\mathcal{N}(\rho) = \sum_i \underset{\substack{\uparrow \\ \text{measure } \rho}}{tr(\rho E_i)}\; \underset{\substack{\uparrow \\ \text{prepare } \sigma_i}}{\sigma_i}$$

Proof: a) $\Rightarrow$ b) $(id_R \otimes \mathcal{N})(\rho_{RA})$ is SEP $\forall \rho_{RA}$, in particular

for γ_{RA} $\quad (|\gamma\rangle_{RA} = \sum_i |i\rangle_R |i\rangle_A)$

b) $\Rightarrow$ c) $\tau_{AB}^{\mathcal{N}}$ is separable: $\exists$ pure states $\psi_i = |\psi_i\rangle\langle\psi_i|_A$

$\varphi_i = |\varphi_i\rangle\langle\varphi_i|_B$ s.t. $\frac{1}{d}\tau_{AB}^{\mathcal{N}} = \sum_i p_i\, \psi_i \otimes \varphi_i$, $d = |A|$

Set $K_i = \sqrt{d p_i}\, |\varphi_i\rangle_B \langle\overline{\psi}_i|_A$

check: $\frac{1}{d}\sum_i (\mathbb{1}_A \otimes K_i)(\gamma_{AA'})(\mathbb{1}_A \otimes K_i)^\dagger$

$$= \frac{1}{d}\sum_{i,j,k} |j\rangle\langle k|_A \otimes K_i |j\rangle\langle k|_{A'} K_i^\dagger$$

$$= \frac{1}{d}\sum_{i,j,k} d\, p_i\, |j\rangle\langle k|_A \otimes \langle\overline{\psi}_i|j\rangle\langle k|\overline{\psi}_i\rangle\, |\varphi_i\rangle\langle\varphi_i|_B$$

$$= \frac{1}{\cancel{d}} \sum_{i,j,h} \cancel{d} p_i \, |j\rangle\langle h|_A \otimes \langle \bar{\psi}_i | j\rangle\langle h | \bar{\psi}_i\rangle \, |\varphi_i\rangle\langle\varphi_i|_B$$

$$= \sum_i p_i \underbrace{\sum_{j,h} \langle h|\bar{\psi}_i|j\rangle \, |j\rangle\langle h|}_{(\bar{\psi}_i)^T = \psi_i^\dagger = \psi_i} \otimes |\varphi_i\rangle\langle\varphi_i|_B$$

$$= \sum_i p_i \, \psi_i \otimes \varphi_i = \frac{1}{d} \tau_{AB}^{N} .$$

$$\sum_i K_i^\dagger K_i = \mathbb{1} = d \sum_i p_i \, |\bar{\psi}_i\rangle \underbrace{\langle\varphi_i|\varphi_i\rangle}_{1} \langle\bar{\psi}_i|$$

$$= d \sum_i p_i \bar{\psi}_i = d \, \overline{(\underbrace{\textstyle\sum_i p_i \psi_i}_{= \overline{\mathrm{tr}_B \frac{1}{d} \tau_{AB}^N} = \bar{\mathbb{1}}_A})} = d \cdot \frac{1}{d} \bar{\mathbb{1}}_A = \mathbb{1}_A \checkmark$$

c) => d) rk $K_i = 1$: $K_i = |\chi_i\rangle_B \langle w_i|_A$ for some vectors

$|\chi_i\rangle_B$, $|w_i\rangle_A$, $\langle\chi_i|\chi_i\rangle = 1$.

$$N(\rho) = \sum_i K_i \rho K_i^\dagger = \sum_i \underbrace{\langle w_i|\rho|w_i\rangle}_{= \mathrm{tr}(\rho w_i)} |\chi_i\rangle\langle\chi_i|_B$$

POVM: $W = \{w_i\}$

states: χ_i

$$\mathbb{1} = \sum_i K_i^\dagger K_i = \sum_i |w_i\rangle \overbrace{\langle\chi_i|\chi_i\rangle}^{=1} \langle w_i|$$

$$= \sum_i |w_i\rangle\langle w_i| \quad \checkmark$$

d) => a) let ρ_{RA} be arbitrary, $\mathcal{N}(\rho) = \sum_i \operatorname{tr}(E_i \rho)\, \sigma_i$

$$(\mathrm{id}_R \otimes \mathcal{N})(\rho_{RA}) = \sum_i \underbrace{\operatorname{tr}_A\left[(\mathbb{1}_R \otimes E_i)\, \rho_{RA}\right]}_{\text{op on } R} \otimes \underset{\text{op on } B}{\sigma_i}$$

1. $E_i \geq 0$

2. cyclicity of tr_A

$$= \sum_i \operatorname{tr}_A\left[(\mathbb{1}_R \otimes \sqrt{E_i})\, \rho_{RA}\, (\mathbb{1}_R \otimes \sqrt{E_i})\right] \otimes \sigma_i$$

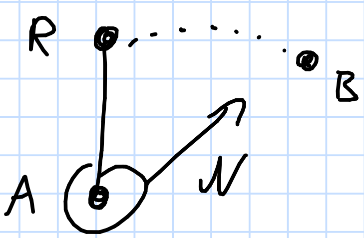
$$= \sum_i p_i\, w_i \otimes \sigma_i$$

where $p_i = \operatorname{tr}(E_i \rho_{RA})$, $w_i = \frac{1}{p_i} \operatorname{tr}_A\left[(\mathbb{1}_R \otimes \sqrt{E_i})\, \rho_{RA}\, (\mathbb{1}_A \otimes \sqrt{E_i})\right] \geq 0$

$\square$

(Some) channel capacities of EB channels are understood:

·) Alice $\xrightarrow{\mathcal{N}}$ Bob quantum information transmission



$\Leftrightarrow$ entanglement generation

but EB channels can't do that

$\Rightarrow$ quantum capacity $Q(\mathcal{N}) = 0$.

·) $C(\mathcal{N}) \geq \chi(\mathcal{N})$, $C(\mathcal{N}) = \cancel{\sup_{n \in \mathbb{N}} \frac{1}{n}}\, \chi(\mathcal{N}^{\otimes n})$ (entangled inputs)

EB channels destroy entanglement between different inputs

$\Rightarrow C(\mathcal{N}) = \chi(\mathcal{N})$ <u>BUT:</u> $\chi(\mathcal{N})$ is NP-hard to compute.

§ 2.4 PPT-channels

checking SEP is NP-hard → some easier criterion?

<u>Peres-Horodecki criterion:</u> ρ_{AB} SEP $\Rightarrow$ ρ_{AB} PPT

PPT: positive partial transpose: $\rho_{AB}^{T_B} \geq 0$

ρ_{AB} SEP: $\rho_{AB} = \sum_i p_i \, \omega_A^i \otimes \sigma_B^i \Rightarrow \rho_{AB}^{T_B} = \sum_i p_i \, \omega_A^i \otimes (\sigma_B^i)^T \geq 0$

·) if ρ_{AB} is NPT $\Rightarrow$ $\rho_{AB} \notin$ SEP

·) if $|A| \cdot |B| \leq 6$, then $\rho_{AB} \in$ SEP $\Leftrightarrow$ $\rho_{AB} \in$ PPT.

Def 11 A channel $\mathcal{N}: A \to B$ is called PPT if

$$(\mathrm{id}_R \otimes \mathcal{N})(\rho_{RA}) \text{ is PPT for all } \rho_{RA}.$$

Prop 12 TFAE:

a) $\mathcal{N}: A \to B$ is PPT

b) $\tau_{AB}^{\mathcal{N}}$ is PPT

c) $\vartheta \circ \mathcal{N}$ is CP

$\uparrow$

$\vartheta: X \mapsto X^T$: $(\mathrm{id}_R \otimes \vartheta)(\gamma) = \mathbb{F} \not\geq 0$