Recap

- Bit- / Phase- / Bitphase flip channels are dephasing channels:

  e.g. $\mathcal{F}_p^Z : \rho \mapsto (1-p)\rho + pZ\rho Z$: $\mathcal{F}_p^Z \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & (1-2p)\rho_{01} \\ (1-2p)\rho_{10} & \rho_{11} \end{pmatrix}$

- $\mathcal{F}_p^Z$ leaves $|0\rangle\langle 0|, |1\rangle\langle 1|$ invariant :

  can send 1 bit of classical information (maximal for qubit chan.)

- Amplitude damping channel A_γ: physical model for energy dissipation in a 2-level system

  Isometry: $V_\gamma : |0\rangle_A \mapsto |0\rangle_B |0\rangle_E$

  $$|1\rangle_A \mapsto \sqrt{1-\gamma}\, |1\rangle_B |0\rangle_E + \sqrt{\gamma}\, |0\rangle_B |1\rangle_E$$

- A_γ is an example of a non-unital channel: $A_\gamma(\mathbb{1}) = \begin{pmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{pmatrix}$

- Quantum capacity of A_γ is known, classical capacity unknown!

- Erasure channel $\mathcal{E}_p$: $\mathcal{B}(\mathbb{C}^2) \longrightarrow \mathcal{B}(\mathbb{C}^2 \oplus \mathbb{C})$

  $$\rho \mapsto (1-p)\rho + p\,\mathrm{tr}(\rho)\, |e\rangle\langle e|$$

  where $\langle e|\rho|e\rangle = 0$ => receiver can always measure whether erasure occurred!

- Generalized dephasing channels: $\mathcal{H} = \mathbb{C}^d$, $\{|i\rangle\}_{i=1}^d$ ONB for $\mathcal{H}$

  $\{|\varphi_i\rangle_E\}_{i=1}^d$ set of arbitrary states: $V : |i\rangle_A \mapsto |i\rangle_B \otimes |\varphi_i\rangle_E$

$V: |i\rangle_A \mapsto |i\rangle_B \otimes |\psi_i\rangle_E \qquad \langle\psi_i|\psi_j\rangle \neq \delta_{ij}$

$$\mathcal{N}(\rho) = \mathrm{tr}_E V\rho V^\dagger = \sum_{i,j} \underbrace{\langle i|\rho|j\rangle \overbrace{\langle\psi_i|\psi_j\rangle}}_{[\mathcal{N}(\rho)]_{ij}} |i\rangle\langle j|_B$$

Generalized dephasing channels have full classical capacity

$$C(\mathcal{N}) = \log d \qquad (\text{in general: } C(\mathcal{N}) \leq \log d)$$

"Proof": We know that, for a gen. dephasing channel, $\exists$ ONB $\{|i\rangle\}_i$

s.t. $\mathcal{N}(|i\rangle\langle i|) = |i\rangle\langle i|$.

d classical signals/messages $x_1, \dots, x_d$

Encoding $x_i \mapsto |i\rangle\langle i|$

Remember: $\langle i|j\rangle = \delta_{ij}$

$\sigma_i = \mathcal{N}(|i\rangle\langle i|) = |i\rangle\langle i| \Rightarrow$ Bob can measure the output to retrieve Alice's classical message.

d messages can be sent perfectly $\Rightarrow$ $\log d$ bits of classical information can be sent through $\mathcal{N}$

$\Rightarrow C(\mathcal{N}) = \log d$. "$\square$"

Detour: Holevo information (χ-quantity)

Recall: von Neumann entropy $S(\rho) = -\operatorname{tr} \rho \log \rho$ $\quad\Big|\quad$ $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, $\quad S(\rho) = -\sum_i \lambda_i \log \lambda_i$

Ensemble of quantum states: $E = \{p_x, \rho_x\}$

$$\chi(E, N) = S\Big(\sum_x p_x N(\rho_x)\Big) - \sum_x p_x S(N(\rho_x))$$

$$\boxed{\text{Holevo information } \chi(N) = \max_E \chi(E, N)}$$

Holevo - Schumacher - Westmoreland - Theorem: classical capacity C satisfies $\boxed{C(N) \geq \chi(N)}$ $\quad \Big(C(N) = \sup_{n \in \mathbb{N}} \frac{1}{n} \chi(N^{\otimes n}) \Big)$

N gen. dephasing channel: $\exists$ ONB $\{|i\rangle\}$ s.t. $N(|i\rangle\langle i|) = |i\rangle\langle i| \ \forall i$

$\underline{\rho_i = |i\rangle\langle i|}$, $p_i = \frac{1}{d}$ $\Rightarrow$ $\bar{\rho} = \sum_i p_i \rho_i = \frac{1}{d}\mathbb{1}$

$E_u = \{\frac{1}{d}, \rho_i\}$: $\quad \chi(E_u, N) = S\Big(\sum_i p_i N(\rho_i)\Big) - \sum_i p_i S(N(\rho_i))$

$N(\rho_i) = \rho_i$

$$= S\Big(\underbrace{\sum_i p_i \rho_i}_{= \bar{\rho} = \frac{1}{d}\mathbb{1}}\Big) - \sum_i p_i \underbrace{S(\rho_i)}_{= 0}$$

$$= \log d - 0 = \log d$$

$\square$!

$$\log d \leq \chi(N) \leq \underline{C(N) \leq \log d} \Rightarrow \boxed{C(N) = \log d}$$

Some comments to last lecture:

·) 50-50 Z-dephasing channel

$$\rho \mapsto \frac{1}{2}\rho + \frac{1}{2} Z\rho Z \quad \Rightarrow \quad K_0 = \frac{1}{\sqrt{2}} \mathbb{1}, \; K_1 = \frac{1}{\sqrt{2}} Z$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} : \quad HXH = Z, \quad H|0\rangle = |+\rangle, \quad H|1\rangle = |-\rangle$$

$$L_i = \sum_j H_{ij} K_j$$

$$\Rightarrow L_0 = |0\rangle\langle 0|, \; L_1 = |1\rangle\langle 1|$$

·) Every unital qubit channel is unitarily equivalent to a Pauli channel: $N: B(\mathbb{C}^2) \to B(\mathbb{C}^2)$ unital channel, then there are unitaries U, V s.t.

$$M(\rho) = U\, N(V \rho V^\dagger) U^\dagger \text{ is Pauli}$$

·) mixed unitary channels $\rho \mapsto \sum_i p_i U_i \rho U_i^\dagger$, U_i unitary

$$\left(\mathbb{1} \mapsto \sum_i p_i U_i U_i^\dagger = \mathbb{1} \quad \text{unital!} \right)$$

Every unital qubit channel is mixed unitary.

$d \geq 3$: example of a unital channel that is not mixed unitary:

$$X \in B(\mathbb{C}^3), \qquad \rho \mapsto \frac{1}{2} \operatorname{tr}(X) \mathbb{1} - \frac{1}{2} X^T$$

[Watrous' book]

§ 2.3 Entanglement-breaking channels

Reminder: A bipartite state ρ_{AB} is called separable, if

$$\rho_{AB} \in \operatorname{conv}\{\, \omega_A \otimes \sigma_B : \omega_A, \sigma_B \text{ states on } \mathcal{H}_{A/B} \,\}$$

Explicitly: $$\underline{\rho_{AB} = \sum_i p_i \, \omega_A^i \otimes \sigma_B^i}$$

no quantum correlation

only classical correlation

Def 9 A channel $\mathcal{N}: A \to B$ entanglement-breaking (EB), if

$(\mathrm{id}_R \otimes \mathcal{N})(\rho_{RA})$ is separable for <u>any</u> ρ_{RA}.

Prop 10 TFAE:

a) $\mathcal{N}: A \to B$ is EB

b) $\tau_{AB}^{\mathcal{N}} = (\mathrm{id}_A \otimes \mathcal{N})(\gamma_{AA'})$ is separable

c) $\mathcal{N}$ has a Kraus rep. where <u>all</u> Kraus op's have rank 1.

d) $\mathcal{N}$ is a measure-and-prepare channel:

$\exists$ POVM $E = \{E_i\}_i$ and states $\{\sigma_i\}_i$ s.t.

$$\mathcal{N}(\rho) = \sum_i \operatorname{tr}(\rho E_i)\, \sigma_i$$